Symbolic dynamics for Kleinian groups

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Thesis submitted for the degree of Doctor of Philosophy at the University of Warwick.

The Mathematics Institute,
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Coventry.

October 1994

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Acknowledgements

I am very grateful to my supervisor Caroline Series for her support and guidance during these four years at Warwick. I thank her for always being available to answer my questions and for introducing me to Hyperbolic Geometry.

I thank Mark Pollicott for his help on the numerous occasions he patiently explained to me zeta functions and Ruelle operators. Not only I was introduced by him to these topics, but also he guided me on how to use them in my research.

I thank John Parker for always being so happy to talk with me about my research problems and for always leaving his office door open to me whenever I wanted to come in and talk.

I thank Elaine, Peta, Fiona, Terry, Mavis, Hazel, Denise and Alice for their help regarding administrative matters and for quite a few happy moments.

I acknowledge the financial support of the Brazilian Government through a grant given by its research council CNPq.

During the years that I was in England together with my family, there was always the invaluable support of my parents and my sister. They are the solid lighthouse that I so often turn to in this sea called life. I am trying my best to serve just as well to my own family.

Finally, there is always Mima, my wife and best friend. Countless times I felt lost during the preparation of this thesis, but our love was always the source of my strength to believe I could succeed, especially when this love we have for each other is so vividly manifested in our two daughters Luisa and Andréa.

Declaration

The results in Chapter 1 constitute a paper which has been accepted for publication in the Mathematical Proceedings, Cambridge Philosophical Society.

The results in Chapter 2 are a joint collaboration with Dr. James W. Anderson from the Department of Mathematics Rice University, Houston USA. Each author equally contributed to the material in this chapter.

The results in the Appendix are due entirely to Dr. Anderson.

The work in this thesis is, to the best of my knowledge, original, except where attributed to others.

Summary

This thesis is composed of three independent chapters and an appendix. In the first two chapters we deal with Kleinian groups and in the third one we concentrate on Fuchsian groups. In chapters 1 and 2, we study the action of a Kleinian group on points in hyperbolic space of three dimensions and on points on its boundary. All Kleinian groups we study have the property that their action on points in the hyperbolic 3-space has a fundamental polyhedron with the property that the union of the images of its faces under the group elements contains each geodesic plane that contains one of the faces. This is the so called *even corners property.* Using this property we prove in Chapter one the existence of an expanding Markov type mapping on the Riemann sphere. The main application is to give a shorter proof that the Selberg zeta function associated to the Kleinian group has a meromorphic extension to $\mathbb{C}$.

In chapter two, we assume the existence of a Kleinian group with the even corner property and define its deformation space. We then prove that the Hausdorff dimension of the limit set of groups in the deformation space varies real analytically as we vary the points in the deformation space. We prove that Klein's combination Theorem quasi-preserves the even corners property in the sense that, if a group Γ is formed by other two via Klein's theorem and these last two are quasi-conformal deformations of groups with the even corners property, then Γ is a deformation of a Kleinian group with the even corners property. In particular, the result about the Hausdorff dimension is valid for all geometrically finite purely loxodromic function groups.

In chapter three we construct an automatic structure for parabolic free Fuchsian groups based on the symbolic coding of points in their limit sets. We then provide a proof, based on their symbolic dynamics, that these groups are automatic. We explicitly determine an automatic structure for the groups.

Chapter 1

Markov partitions and zeta functions

We prove the existence of a piecewise analytic expanding map associated to certain Kleinian groups without parabolics acting on points in the 3-dimensional hyperbolic space. These groups have a fundamental domain $\mathcal{R}$ with the the property that the geodesic planes containing each face are part of the tessela-tion. We use this map together with the methods of thermodynamic formalism to give another proof that the Selberg zeta function for such groups has a meromorphic extension to $\mathbb{C}$

1.1. Introduction

The aim of this chapter is to show the existence of a piecewise analytic expanding map defined on the limit set of a Kleinian group having a fundamental domain $\mathcal{R}$ with the so called *even corners* property. This is a condition on the geodesic planes that contain each face of $\mathcal{R}$. Following the work of C. Series as described in [Ser91], we use the face pairing identifications to define a Markov type transformation associated to a partition of the limit set of the group. If the Kleinian group has no parabolics, we prove that this map is (eventually) expanding. In line with the works of Ruelle [Rue76], Pollicott [Pol91] and Mayer [May91b] we use this mapping to define Ruelle type operators, and using Grothendieck's theory of nuclear operators, see [Gro56], we express a specific Ruelle zeta function in terms of the determinants of these operators. The objective of these steps is to relate the Ruelle zeta function, which we know to have a meromorphic extension to the whole complex plane, to the Selberg zeta function for the Kleinian group. We then are able to provide a short proof that the Selberg zeta function for such groups has a meromorphic extension to $\mathbb{C}$.

1.2. Definitions and preliminary results

A Kleinian group is a discrete subgroup of the group of orientation preserving isometries of the three dimensional hyperbolic space $\mathbb{H}^3$. We use the Poincaré ball model to realize hyperbolic space in three dimensions. Recall that this model consists of the unit ball in $\mathbb{R}^3$

$$\mathbb{B}^3 = \{x \in \mathbb{R}^3 : |x| < 1\}$$

with the (hyperbolic) metric

$$ds = \frac{2dr}{1 - |r|^2}.$$

Let Γ be a finitely generated non-elementary Kleinian group acting on $\mathbb{H}^3$. The limit set Λ of Γ is the set of limit points of the orbit under Γ of any point $x \in \mathbb{H}^3$. Because Γ acts discontinuously, clearly we have that $\Lambda \subset S^2$.

The elements of Γ belong to the subgroup of all Möbius transformations that preserve $\mathbb{B}^3$, which is in turn a subgroup of the full Möbius group $\mathbf{Mob}_3$, see [Ahl81]. The elements of $\mathbf{Mob}_3$ are real analytic bijective maps in a suitable open set in $\mathbb{R}^3$ which we will deal with later.

We assume that Γ has no parabolic elements.

Definition 1.2.1 (Even corners property). A fundamental domain $\mathcal{R}$ for Γ is said to have the *even corners property* if

$$\bigcup_{\gamma \in \Gamma} \gamma(\partial \mathcal{R})$$

is a union of planes.

We assume that $\mathcal{R}$ is a fundamental polyhedron for Γ with the above property and such that $0 \in \mathcal{R}$.

Let $\Gamma_{\mathcal{R}}$ be the set of face pairing transformations for $\mathcal{R}$. Using Poincaré's Theorem, we get a presentation $< \Gamma_{\mathcal{R}} | R_{\mathcal{R}} >$ for the group Γ, such that $\Gamma_{\mathcal{R}}$ is a symmetric system of generators and $R_{\mathcal{R}}$ is the set of relations we get from each vertex and face cycles, see [Mas88]. We use the elements in $\Gamma_{\mathcal{R}}$ to label the faces of $\mathcal{R}$ as follows. Use the label e on a face of $\mathcal{R}$ if the element $e \in \Gamma$ pairs it to some other face. Write e inside $\mathcal{R}$ and $\overline{e}$ on the outside, where $\overline{e}$ denotes $e^{-1} \in \Gamma_{\mathcal{R}}$.

Following Bourdon [Bou93], we have for each word $e_{i_0}e_{i_1} \cdots e_{i_n}$ written in the generators in $\Gamma_{\mathcal{R}}$ a family of planes $P(e_{i_0}), P(e_{i_0}e_{i_1}), \ldots, P(e_{i_0} \ldots e_{i_n})$

defined as : $P(e_{i_0})$ is the plane containing the face $\mathcal{R} \cap e_{i_0}\mathcal{R}$ and for $k = 1, \ldots, n$ $P(e_{i_0}e_{i_1}\ldots e_{i_k})$ is the plane containing $e_{i_0}\ldots e_{i_{k-1}}\mathcal{R} \cap e_{i_0}\ldots e_{i_k}\mathcal{R}$.

Proposition 1.2.2. *The following statements are equivalent :*

(a) *The word $e_{i_0}\ldots e_{i_n}$ is shortest in $\Gamma_{\mathcal{R}}$*

(b) *For $k = 0, \ldots, n$ the planes $P(e_{i_0}\ldots e_{i_k})$ are pairwise distinct.*

(c) *For every $k \in \{0, \ldots, n\}$, and for every l satisfying $k \leq l \leq n$, the plane $P(e_{i_0}\ldots e_{i_k})$ separates $\mathcal{R}$ from $e_{i_0}\ldots e_{i_l}\mathcal{R}$.*

Proof of 1.2.2.

This is in Bourdon [Bou93], §3.3. In fact he proved this result for any dimension. $\boxed{1.2.2}$

We introduce a few more definitions and prove a theorem that we will use later. Let $H(e)$ be the half-space in $\mathbb{B}^3$ whose intersection with $\mathcal{R}$ is the face whose exterior label is e. It is clear that $\{H(e_i) : e_i \in \Gamma_{\mathcal{R}}\}$ and $\mathcal{R}$ have union $\mathbb{B}^3$. Since $H(e_i)$ contains a face of $\mathcal{R}$, we could use the associated face pairing to define a map f that sends a point x in $H(e_i)$ to $e_i^{-1}x$ if i is the least element of $\{j : x \in H(e_j)\}$ and $f(x) = x$ if $x \in \mathcal{R}$. We define :

$$
\begin{aligned}
B(e_i) &= \{x \in H(e_i) : f(x) = e_i^{-1}x\}, \\
I(e_i) &= S^2 \cap \overline{B(e_i)}, \\
B(e_0, \ldots, e_n) &= \cap_{r=0}^{n} f^{-r} B(e_r), \\
I(e_0, \ldots, e_n) &= S^2 \cap \overline{B(e_o, \ldots, e_n)} \\
&= \cap_{r=0}^{n} f^{-r} I(e_r).
\end{aligned}
$$

Theorem 1.2.3.

(a) *Int $B(e_0, \cdots, e_n) \neq \varnothing \Leftrightarrow$ Int $I(e_0 \cdots e_n) \neq \varnothing$.*

(b) *Int $B(e_0 \cdots e_n) \neq \varnothing \Rightarrow e_0 \cdots e_n$ is shortest.*

(c) *If $f(g \cdot \mathcal{R}) = h \cdot \mathcal{R}$, then $|h| = |g| - 1$ ($|\cdot|$ denotes the word length norm with respect to the generating set $\Gamma_{\mathcal{R}}$).*

(d) *If Int $B(e_0, \cdots, e_n) \neq \varnothing$, then diam $I(e_0 \cdots e_n) \to 0$ as $n \to \infty$.*

Proof of 1.2.3.

The proof of this theorem for the two dimensional case is due to C. Series, and can be found in [Ser91]. Proposition 1.2.2 allows us to extend her proof of (a),(b),(c) to the present case. We prove that (d) is also true in the present context. Observe that $B(e_0e_1) = B(e_0) \cap e_0B(e_1)$, and then

$$f^2|_{B(e_0e_1)} = e_1^{-1}e_0^{-1}.$$

Use induction to get

$$f^n|_{B(e_0e_1\cdots e_{n-1})} = e_{n-1}^{-1}\cdots e_0^{-1}$$

and now it is easy to see that

$$B(e_0\cdots e_n) = e_0\cdots e_{n-1}B(e_n) \cap B(e_0\cdots e_{n-1}).$$

From this last equality we have $B(e_0e_1\cdots e_n) \subset e_0e_1\cdots e_{n-1}H(e_n)$ and then

1.2.4. $$I(e_0e_1\cdots e_n) \subset e_0e_1\cdots e_{n-1}H(e_n).$$

If $x, y \in \mathbb{B}^3$, then denote their hyperbolic and Euclidean distance by $d_H(x,y)$ and $d_E(x,y)$, respectively. Recall two important estimates relating the hyperbolic distance and the word metric :

(i) If Γ has no parabolics, then there exists a constant $\alpha > 0$ such that if $g \in \Gamma - \{id\}$ then $d_H(0, g0) > \alpha|g|$.

(ii) If Γ has parabolics, then there exist constants $k, n_0 \in \mathbb{N}$ such that if $g \in \Gamma - \{id\}$ then $d_H(0, g0) > 2\log|g| - k$ if $|g| > n_0$.

Proofs of these inequalities can be found in [Flo80]. Let $h \in \Gamma - \{id\}$ be such that $d_H(0, h0) > L$, for some $L > 0$. If $B_H(h0, r)$ is a hyperbolic ball with centre $h0$ and hyperbolic radius r, $L > r$, then it is well known that

1.2.5. $$B_H(h0, r) \subseteq B_E(h0, \varepsilon),$$

where $B_E(h0, \varepsilon)$ is an Euclidean ball of radius ε with centre $h0$ and Euclidean radius

$$\varepsilon = \frac{1}{2}r\cosh^{-2}\left(\frac{(L-r)}{2}\right) \leq c_0e^{-L},$$

where c_0 is a constant which does not depend on L, but does depend on r.

Let $w = w_1w_2\cdots w_N$ and $v = v_1v_2\cdots v_M$, $M > N$ with $w_1 = v_1, \ldots, w_n = v_n$ for some $n \leq N$, and such that $B(v_1v_2\cdots v_M)$ and $B(w_1w_2\cdots w_N)$ are non-empty. Let $r = \max\{d_H(0, e0) : e \in \Gamma_{\mathcal{R}}\}$. If Γ has no parabolics, then

1.2.6. $$d_E(w0, v0) \leq \sum_{i=0}^{N-1} d_E(w_1\cdots w_{n+i}0, w_1\cdots w_{n+i+1}0)$$

1.2.7. $$+\sum_{i=0}^{M-1} d_E(v_1\cdots v_{n+i}0, v_1\cdots v_{n+i+1}0).$$

Using (i) above,

$$d_H(0, v_1 \cdots v_{n+i}) > \alpha(n+i)$$

and therefore

$$B_H(w_1 \cdots w_{n+i}0, r) \subseteq B_E(w_1 \cdots w_{n+i}0, \beta),$$

where $\beta = c_0 \exp(-\alpha(n+i))$. Since $d_H(w_1 \cdots w_{n+i}0, w_1 \cdots w_{n+i+1}0) \leq r$,

$$d_E(w_1 \cdots w_{n+i}0, w_1 \cdots w_{n+i+1}0) \leq c_0 \exp(-\alpha(n+i)).$$

We can obtain a similar relation for v, and then it follows that

$$d_E(w0, v0) \leq \gamma \exp(-\alpha n).$$

If Γ has parabolics, then the same argument works with (b) instead of (a), it then follows that

1.2.8.
$$d_E(w0, v0) \leq 2c \sum_{i=0}^{\infty} \frac{1}{(n+i)^2}.$$

These estimates were proved by C. Series in [Ser81]. Consider the sequence $\{g_i 0\}_{i=0}^{\infty}$ where $g_i = e_0 e_1 \cdots e_i$ with $B(e_0 e_1 \cdots e_i) \neq \varnothing$ for all $i = 0, 1, \ldots$. This sequence converges to a point in the limit set Λ of Γ. Now two observations to finish our argument. Clearly $g_0 0 \in H(e_0)$ and for each $i \geq 1$ we have $g_i 0 \in e_0 e_1 \cdots e_{i-1} H(e_i)$, also the half-spaces $H(e_0), e_0 H(e_1), \ldots$ form a nested sequence of sets, that is to say

$$H(e_0) \supseteq e_0 H(e_1) \supseteq e_0 e_1 H(e_2) \supseteq \ldots e_0 e_1 \cdots e_{n-1} H(e_n) \supseteq \ldots.$$

Recall that each of these half-spaces is a complete union of copies of the fundamental domain $\mathcal{R}$ due to the fact that $\mathcal{R}$ is even cornered. Combining these conclusions together with estimates 1.2.6 and 1.2.8 we have that

1.2.9.
$$diam\, e_0 e_1 \cdots e_{n-1} H(e_n) \overset{n \to \infty}{\longrightarrow} 0,$$

otherwise points in the planes $P(e_0 \ldots e_n)$ which bound each half-space would accumulate and this contradicts the discreteness of Γ.

$\boxed{1.2.3}$

1.3. A subshift of finite type

Consider the set $\mathfrak{P}$ of all planes in $\Gamma \cdot \partial\mathcal{R}$ that intersect $\mathcal{R}$ in either a face, an edge or a vertex. The intersection of each plane in $\mathfrak{P}$ with S^2 is a circle, and the closures of the components of the complement of the union of these circles cover S^2. Although these regions can intersect their interiors are disjoint. This gives us either a finite or countable partition $\mathfrak{R} = \{R_i\}_{i=1}^{\infty}$ of S^2 minus a set of Lebesgue measure zero.

The map f was defined from $\mathbb{B}^3$ to itself using the half-spaces $H(e_i)$ and it is not difficult to see that it extends to S^2 in the same way it was defined. We would like to use the partition $\mathfrak{R}$ to prove the Markov property for f, that is to say, the image under f of any element $R_i \in \mathfrak{R}$ is a union of elements in $\mathfrak{R}$. To do so, we assume that $\mathfrak{R}$ partitions the entire S^2 and then we have that

Proposition 1.3.1. *The map f has the Markov property with respect to the partition $\mathfrak{R} = \{R_i\}_{i=1}^{\infty}$.*

Proof of 1.3.1.

Let $R_i \in \mathfrak{R}$. The boundary of R_i is a union of arcs belonging to circles which are the intersection of planes in $\mathfrak{P}$ with S^2. From the definition of the half-spaces $H(e_i)$, it follows that R_i is contained in $H(e_j)$, some j such that $f|_{R_i} = e_j^{-1}$. Suppose that e_j^{-1} pairs the face F_j to the face F_{i_j}. Observe that e_j^{-1} sends the planes in $\mathfrak{P}$ intersecting F_j to similar ones intersecting F_{i_j}. Therefore, the circles containing the arcs of the boundary of R_i are sent to circles which are the intersections of the images of the planes passing through F_j with S^2. Therefore, the boundary of the image of R_i is a union of arcs in distinct circles, this implies that the region R_i is mapped under e_j^{-1} to a union of regions in $\mathfrak{R}$. $\boxed{1.3.1}$

Following the standard procedures of symbolic dynamics we now introduce a subshift of finite type on the space

$$\Sigma_f = \{(r_k)_{k=0}^{\infty} : f(R_{r_k}) \supseteq (R_{r_{k+1}}),\ r = 0, 1; \ldots\}$$

by $\sigma : \Sigma_f \to \Sigma_f$, $\sigma(r_k)_{k=0}^{\infty} = (r_{k+1})_{k=0}^{\infty}$. We refer the reader to [PP90] for the general theory of subshifts of finite type.

Consider the following set

$$\Omega = \{(e_i)_{i=0}^{\infty} : B(e_0 e_1 \cdots e_i) \neq \varnothing, e_i \in \Gamma_{\mathcal{R}},\ i = 0, 1, \ldots\}.$$

Clearly the sets Σ_f and Ω are related. Indeed, for each r_k define $\psi(r_k) = e \in \Gamma_{\mathcal{R}}$ if $f|_{R_k} = e^{-1}$. The Markov property of f guarantees that the map

$$\psi : \Sigma_f \to \Omega$$

defined by $\psi((r_k)_{k=0}^{\infty}) = (\psi(r_k))_{k=0}^{\infty}$ is well defined. This map allows us to define a mapping from the shift space Σ_f to the limit set Λ of Γ. Using the fact that

$$I(e_0 e_1 \cdots e_n) \supseteq I(e_0 e_1 \cdots e_{n+1})$$

and *diam* $I(e_0 e_1 \cdots e_n) \to 0$ as $n \to \infty$ then, clearly if $\underline{r} = (r_0, r_1, \cdots) \in \Sigma_f$,

$$\pi(\underline{r}) = \bigcap_{k=0}^{\infty} I(\psi(r_0)\psi(r_1) \cdots \psi(r_k))$$

is a single point in $I(\psi(r_0)) \cap \Lambda$. Observe that points on the boundary of each R_k can have more than one f-expansion, and therefore the map $\pi : \Sigma_f \to \Lambda$ possibly fails to be a bijection precisely at these points. Therefore we have

Theorem 1.3.2. *There is a continuous map π between the subshift of finite type Σ_f and the limit set Λ of Γ which is bijective, except in a subset $\Sigma' \subset \Sigma_f$, which is mapped under π in a set of zero Lebesgue measure.*

Observe that $f\pi(\underline{r}) = \pi(\sigma(\underline{r}))$.

1.4. An expanding map

From this section onwards, the group Γ will be considered *without parabolics.* We review some basics facts about Möbius transformations in $\mathbb{R}^3$ as described in Ahlfors [Ahl81].

Definition 1.4.1 (similarities and reflection in S^2). The group of similarities are all mappings $x \mapsto mx + b$ where $b \in \mathbb{R}^3, m = \lambda k$, $\lambda > 0$ and $k \in O(3)$, in words, m is a conformal matrix. Reflection, or inversion, with respect to the unit sphere is defined by $x \mapsto x^* = \frac{x}{|x|^2}$, where $x = (x_1, x_2, x_3)$ and $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Definition 1.4.2 (Möbius group). The full Möbius group $\mathbf{Mob}_3$ is the group generated by all similarities together with the inversion in the unit sphere. The Möbius group $\mathbf{Mob}_3^+$ is the subgroup whose elements are products of an even number of inversions in the unit sphere and similarities which preserve orientation.

We are interested in those Möbius transformations in $\mathbf{Mob}_3^+$ which preserve $\mathbb{B}^3$, and since Möbius mappings are bijective the unit sphere S^2 and the exterior of $\mathbb{B}^3$ are also preserved. Needless to say that the elements of Γ are in this class. We call $\mathbf{Isom}(\mathbb{B}^3)$ the subgroup of $\mathbf{Mob}_3^+$ which keeps $\mathbb{B}^3$ invariant.

In order to get a more explicit description of those transformations in $\mathbf{Isom}(\mathbb{B}^3)$, we quote some formulae for $\gamma \in \mathbf{Isom}(\mathbb{B}^3)$ and its derivative as well. Once again, proofs for these formulae can be found in Ahlfors [Ahl81]. Given $a \in \mathbb{B}^3$ let T_a be the following Möbius transformation

$$T_a(x) = \frac{(1-|a|^2)(x-a) - |x-a|^2 a}{[x,a]^2},$$

where $[x,a] = |x||x^* - a| = |a||x - a^*|$ and $[x,a]^2 = 1 + |x|^2|a|^2 - 2x \cdot a$, where $x \cdot a$ denotes the inner product of a and x.

The norm of the derivative of a differentiable map f from $\mathbb{R}^n$ to $\mathbb{R}^m$, which we denote by $\|D_x f\|$, is the supremum norm of the Jacobian matrix $(f'(x)_{ij})$ where $f'(x)_{ij} = \frac{\partial f_i}{\partial x_j}$.

Proposition 1.4.3. *Let $\gamma \in \mathbf{Isom}(\mathbb{B}^3)$. Then it can be written uniquely as*

1.4.4.
$$\gamma = kT_a$$

where $k \in O(3)$ and $a = \gamma^{-1}0$. The norm of the derivative of γ written as above is

1.4.5.
$$\|D_x\gamma\| = \|D_x T_a\| = \frac{1-|a|^2}{[x,a]^2}$$

We call 1.4.4 *the* canonical representation *of $\gamma \in \mathbf{Isom}(\mathbb{B}^3)$*

Proof of 1.4.3.

Again, see Ahlfors [Ahl81] $\boxed{1.4.3}$

Remark 1.4.6. It is worthwhile to mention that the above transformation T_a is not defined at the point a^* if we are not considering the one point compactification of $\mathbb{R}^3$. A more important fact for us is that from expression 1.4.5, we see that the derivative of γ is always non-zero.

We assume that from now on every element in Γ is expressed in its canonical representation. The immediate conclusion is that every element in the Kleinian group Γ is a real analytic mapping. It is this important fact that is behind the ideas in what follows next.

Theorem 1.4.7. *The map f is (eventually) expanding.*

Proof of 1.4.7.

We use some of C. Series' ideas exposed in [Ser81]. For each $n \geq 0$ define the following subset of Σ_f

$$\Sigma_n = \{(r_k)_{k=0}^n \in \Sigma_f : \lambda\left(\cap_{i=0}^n f^{-i}(R_{r_i})\right) > 0\},$$

where λ is the Lebesgue measure on S^2. For $(r_0 r_1 \dots r_n) \in \Sigma_n$ define $R(r_0 r_1 \dots r_n) = \cap_{i=0}^n f^{-i}(R_{r_i})$. Let $T : R(r_0 r_1 \dots r_n) \to R(r_n)$ be given by

1.4.8. $$T = f^n|_{R(r_0 r_1 \dots r_n)} = \psi(r_{n-1})^{-1} \cdots \psi(r_0)^{-1}.$$

Note that T is a bijection and also a Möbius transformation. Use Proposition 1.4.3 to write T in its canonical form $T = kT_{a_n}$, where $a_n = T^{-1}0 = \psi(r_0) \cdots \psi(r_{n-1})0$. Still from Proposition 1.4.3 we obtain

1.4.9. $$\frac{\|D_y T\|}{\|D_x T\|} = \frac{|x - a_n|^2}{|y - a_n|^2}$$

for $x, y \in R(r_0 \dots r_n)$. We want to show that 1.4.9 is bounded by a constant which does not depend on x, y and also is independent of the number of iterations taken in the definition of T, see 1.4.8. Recall that the boundary of the half-space $\psi(r_0) \cdots \psi(r_{n-1}) H(\psi(r_n))$ is the plane $\psi(r_0) \cdots \psi(r_{n-1}) P(\psi(r_n))$, and this last one is the intersection of a sphere perpendicular to S^2 with $\mathbb{B}^3$. Denote by $S(\psi(r_0) \dots \psi(r_n))$ the intersection of $\psi(r_0) \cdots \psi(r_{n-1}) P(\psi(r_n))$ with S^2 and let N_n be the point in this plane which is equidistant, with respect to the Euclidean metric in $\mathbb{R}^3$, to the points in $S(\psi(r_0) \cdots \psi(r_n))$, see figure 1. It is clear that $S(\psi(r_0) \dots \psi(r_n))$ is a circle which divides S^2 in two regions, call $C(r_0 \cdots r_n)$ the one containing $R(r_0 \dots r_n)$. For $n = 1, 2, \dots$ define

$$c_n = \sup\left\{\frac{|x - a_n|}{|y - a_n|} : x, y \in C(r_0 \dots r_n)\right\}.$$

Since $C(r_0 \dots r_n)$ is compact $c_n = |x_n - a_n|/|y_n - a_n|$ for some $x_n, y_n \in C(r_0 \dots r_n)$, and moreover, using elementary Euclidean geometry it is not hard to see that $x_n \in S(\psi(r_0) \dots \psi(r_n))$ and that y_0 is the intersection of the line joining 0 to a_n with $C(r_0 \dots r_n)$, see figure 1 below.

In order to get a better understanding of the claims we are about to make and prove, we refer the reader to Figure 1. Let x_n' be the point in $S(\psi(r_0) \dots \psi(r_n))$ opposite to x_n. We denote by $\overline{a_n x_n}$, $\overline{x_n' x_n}$ and $\overline{N_n x_n}$ the

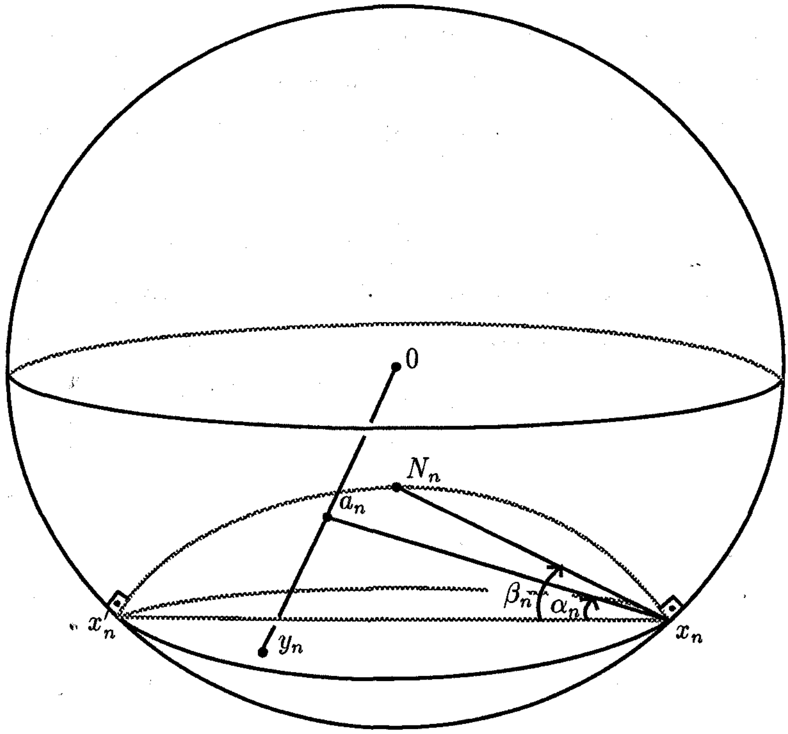

Figure 1.1. The typical situation

segments connecting a_n to x_n, x'_n to x_n and N_n to x_n respectively. Let β_n be the angle between $\overline{x'_n x_n}$ and $\overline{N_n x_n}$ and α_n the one between $\overline{a_n x_n}$ and $\overline{x'_n x_n}$. Let $d_H(a_n, N_n)$ denote the hyperbolic distance between the points a_n and N_n defined above. We have that

Lemma 1.4.10. *$d_H(a_n, N_n) \leq C$, where C is a constant which does not depend on n.*

Proof of 1.4.10.

The argument given here to prove the Lemma was suggested to me by Dr. Martin Bridson.

Since the group Γ is geometrically finite and has no parabolics, we have that the embedding of its Cayley graph in $\mathbb{H}^3$ is a quasi-isometry. A proof of this fact can be found in [Flo80] or in [Can84]. It then follows that there exists a constant C, which depends only on the generator set $\Gamma_{\mathcal{R}}$, such that a geodesic in the Cayley graph is C-close with respect to the hyperbolic metric to the geodesic in $\mathbb{H}^3$ with the same end points.

Let λ_n be the geodesic in $\mathbb{H}^3$ connecting 0 to N_n. Observe that a_n lies in a geodesic in the Cayley graph connecting 0 to the end point of λ_n lying in

$C(r_0 \dots r_n)$ because this end point is coded in the shift Σ_f as $(r_0 r_1 \cdots r_n \cdots)$. Then we can say that the two geodesics are at most C hyperbolic distance apart using what was said in the last paragraph. The Lemma now follows.

$\boxed{1.4.10}$

Using the above lemma we are able to say that as n goes to infinity we must have $\lim_{n\to\infty} |\alpha_n - \beta_n| = 0$. We have already proved that the diameter of $S(\psi(r_0)\dots\psi(r_n))$ goes to zero as n increases. This result and the fact that the planes $\psi(r_0)\dots\psi(r_{n-1})P(\psi(r_n))$ are perpendicular to S^2 imply that $\lim_{n\to\infty} |\beta_n - \frac{\pi}{4}| = 0$. Thus, for some n_0 we have that

$$\alpha_n > \frac{\pi}{6}$$

for all $n \geq n_0$. Let

$$k_1 = \min\left\{\sin\frac{\pi}{6}, \sin\alpha_1, \dots, \sin\alpha_{n_0-1}\right\}.$$

Consider the triangle Δ with vertices at x_n, a_n^p and a_n, where a_n^p is the orthogonal projection of the point a_n onto the plane containing $S(\psi(r_0)\dots\psi(r_n))$, see Figure 2.

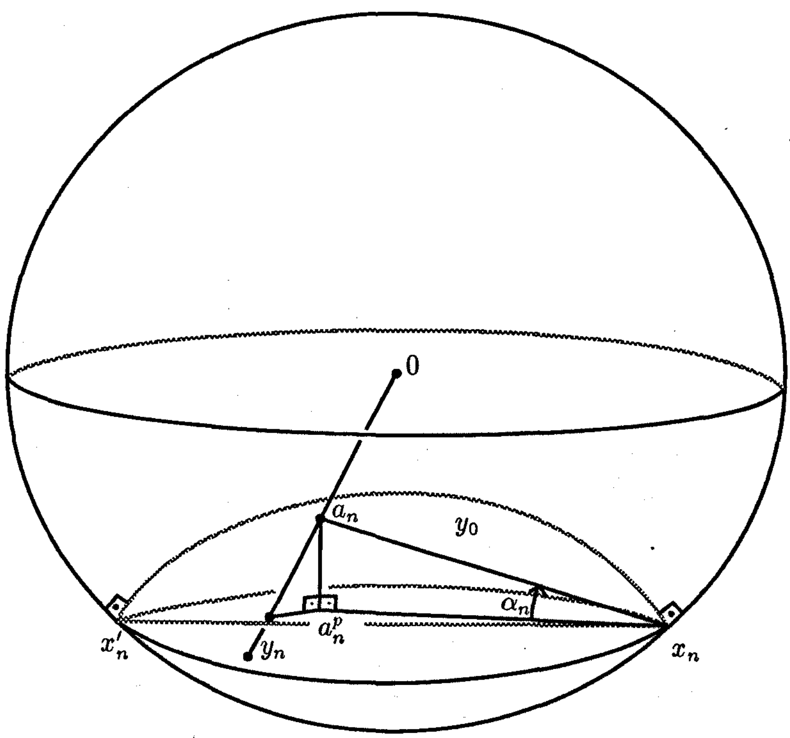

Figure 1.2. The triangle Δ with vertices at x_n,a_n^p and a_n

We have
$$\sin \alpha_n = \frac{|a_n - a_n^p|}{|a_n - x_n|}.$$
Since $|a_n - a_n^p| < |a_n - y_n|$ we get $|y_n - a_n| > k_1|x_n - a_n|$ for all $n = 1, 2, \ldots$. It is not difficult to see that
$$\frac{|x_n - a_n|}{|y_n - a_n|} \leq \frac{3}{k_1}.$$

Finally, we obtain

1.4.11.
$$\left(\frac{k_1}{3}\right)^2 \leq \frac{\|D_x T\|}{\|D_y T\|} \leq \left(\frac{3}{k_1}\right)^2$$

for all $x, y \in R(r_0, \ldots r_n)$.

The change of variable formula gives us
$$\begin{aligned} \lambda(R(r_n)) &= \int_{R(r_0 r_1 \ldots r_n)} |\det D_x T| dx \\ &\leq k_2 M (diam\, R(r_0 r_1 \ldots r_n))^2, \end{aligned}$$
where k_2 is a constant and
$$M = \sup \{\|D_x T\| : x \in R(r_0 r_1 \ldots r_n)\}.$$
Using 1.4.11 and this last inequality we get
$$\|D_x T\| \geq \frac{k_3}{(diam\, R(r_0 r_1 \ldots r_n))^2},$$
with k_3 another constant. Thus, choosing n such that $diam\, R(r_0 r_1 \ldots r_n)) < \sqrt{k_3}$ proves the result.

$\boxed{1.4.7}$

1.5. Application: The Selberg Zeta Function

In line with the results of Pollicott [Pol91] for zeta function associated to compact Riemann surfaces, we apply here some of the methods in thermodynamic formalism to give a short proof that the Selberg zeta function for the groups we consider in this work has a meromorphic extension to the whole complex plane $\mathbb{C}$.

In this section we realize hyperbolic space in three dimensions using the upper-half space model which is defined as the set $\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 : t > 0\}$ equipped with the metric
$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}.$$

This model will prove to be more convenient for the purposes of this section. Recall that the boundary at infinity of $\mathbb{H}^3$ is $\overline{\mathbb{C}}$, the one point compactification of the complex plane $\mathbb{C}$. We get a conformal homeomorphism from S^2 to $\overline{\mathbb{C}}$ using stereographic projection $q : S^2 \to \overline{\mathbb{C}}$.

The group of orientation preserving isometries of $\mathbb{H}^3$ is $PSL(2,\mathbb{C})$. This group acts on points belonging to $\partial\mathbb{H}^3$ via fractional linear transformations. In this context, a Kleinian group is a discrete subgroup of $PSL(2,\mathbb{C})$. Let Γ be a Kleinian group acting freely in $\mathbb{H}^3$. The quotient space $M(\Gamma) = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold and the natural projection $\pi : \mathbb{H}^3 \to M(\Gamma)$ is a covering map. Recall that an element of $PSL(2,\mathbb{C})$ is called *loxodromic* if it has two fixed points on $\overline{\mathbb{C}}$. The axis of a loxodromic element is the geodesic in $\mathbb{H}^3$ whose end points are the two fixed points. Every closed geodesic $\gamma \in M(\Gamma)$ is the projection under π of the axis of a loxodromic element in Γ. The length $l(\gamma)$ of γ is the displacement distance of the loxodromic element whose image under π is γ. The set of all closed geodesics is countable.

Any loxodromic element can be conjugated to one which fixes 0 and ∞, that is to say, to an element whose action on $z \in \overline{\mathbb{C}}$ is given by $z \mapsto \lambda e^{i\theta} z$, $\lambda > 0$. The magnification factor λ and the angle θ are conjugacy invariants associated to the geodesic γ.

The Selberg zeta function of Γ is defined as, see [Fri86]

1.5.1.
$$Z_{\text{Selberg}}(s) = \prod_{\gamma} \prod_{k=0}^{\infty} \left(1 - e^{-i\theta k} e^{(s+k)l(\gamma)}\right) \left(1 - e^{i\theta k} e^{(s+k)l(\gamma)}\right).$$

where γ runs through all closed oriented geodesics of prime period, including their reverse. The product converges for $\Re(s) > 2$.

We want to prove that $Z_{\text{Selberg}}(s)$ has a meromorphic extension to $\mathbb{C}$ using the methods developed by Ruelle in [Rue76]. The first step in this direction was to construct the expanding map f defined in last section. We want to define a partition and a mapping with the same properties but this time we work on $\overline{\mathbb{C}}$. We get a Markov partition for $\overline{\mathbb{C}}$ by mapping each element in the partition we defined for S^2 via the stereographic projection q. We denote this partition of $\overline{\mathbb{C}}$ by $\mathfrak{R}$. In order to get a Markov map on $\overline{\mathbb{C}}$ we use the mapping f as defined in the last section. Conjugate it by q to get the map $q \circ f \circ q^{-1}$ which we still denote by f. It follows from Theorem 1.4.7 that f is expanding. Note that f is now given by a fractional linear transformation on each element of the partition.

If x is a fixed point of f^n, $n \geq 1$, then x must be the end point of a geodesic in $\mathbb{H}^3$ which projects to a closed geodesic in $\mathbb{H}^3/\Gamma$. Conversely, the f-expansion of an end point x of a closed geodesic in $\mathbb{H}^3/\Gamma$ (we mean the end point of one of its lift) must be periodic, and therefore $f^n(x) = x$, for some n . Thus, there is a one-to-one correspondence between closed geodesics in $\mathbb{H}^3/\Gamma$ and fixed points of $f^n, n \geq 1$. It is straightforward to verify that

1.5.2. $$l(\gamma) + i\theta = \log |(f^n)'(x)| + i \arg((f^n)'(x)).$$

Indeed, suppose that we have $f^n(x) = x$ for some $n \geq 1$. From what we said in the last paragraph, f^n is a fractional linear transformation g given by $g(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Observe that

$$g'(x) = \frac{1}{\left(\frac{trace(g)-(trace^2(g)-4)^{\frac{1}{2}}}{2}\right)^2}$$

and since $trace(g) = e^{\lambda/2+i\theta/2} + e^{-\lambda/2-i\theta/2}$, we get the the formula given in 1.5.2

We now follow the approach adopted in [Pol94] and in [May91b]. Consider the partition $\mathfrak{R}$ as defined above. Because Γ has no parabolics, the partition $\mathfrak{R}$ has only finitely many elements. In order to fix our notation, let $\mathfrak{R} = \{R_1, \ldots, R_m\}$ be the elements of the partition. For each $R_i \in \mathfrak{R}$ we define an open convex neighbourhood $U_i \supseteq R_i$, such that $f|_{R_i}$ extends to a real analytic function in U_i (we can assume for $int\, R_i \subseteq int\, R_j$ we have $f^{-1}U_j \subseteq U_i$).

Denote by $D = \coprod_{i=1}^m U_i$ the disjoint union of the U_i's. Following Ruelle [Rue76], we define $B_0(D)$ as the Banach space of those analytic function in D with a continuous extension to the closure $\overline{V_i}$ of each V_i, where $V_i \subseteq \mathbb{C}^2$ is an open neighbourhood of U_i. Denote the extension by f. Since $f|_{U_i}$ is a fractional linear transformation, each of its local inverses f_i belongs to $B_0(D)$. Also, let $B_1(D)$ and $B_2(D)$ be the spaces of real analytic exterior forms of order 1 and 2 respectively, whose coefficients have a continuous extension to the closure of each V_i.

Motivated by formula 1.5.2, we define for each integer $k \geq 0$ and for fixed $s \in \mathbb{C}$ the operator $L_{s,0}^{(k)} : B_0(D) \to B_0(D)$ by

$$(L_{s,0}^{(k)} h)(z) = \sum_{\{j: z \in V_i \text{ and } f(V_j) \supseteq V_i\}} \exp(-s \log |f_j'(z)| + ik \arg(f_j'(z)))h(f_j(z)),$$

$z \in \mathbb{C}^2$. Likewise, we define the operators $L_{s,1}^{(k)} : B_1(D) \to B_1(D)$ and $L_{s,2}^{(k)} : B_2(D) \to B_2(D)$.

We remind the reader of the definition of *nuclear operator*. Let B be a Banach space. An operator $T : B \to B$ is called *nuclear*, if for all $x \in B$ we can write $Tx = \sum_{n=0}^{\infty} \alpha_n b_n^*(x) b_n$, where $b_n \in B$, $b_n^* \in B^*$ are such that $\|b_n\| = \|b_n^*\| = 1$ for all $n \geq 0$ and $(\alpha_n)_{n=0}^{\infty} \in l_1$. The reader should consult Grothendieck [Gro56] for full proofs of the results on nuclear operators that we use from now on.

Once we know the operators $L_{s,0}^{(k)}$, $L_{s,1}^{(k)}$ and $L_{s,2}^{(k)}$ are nuclear of order zero, we can calculate their traces. These traces are related to the partition function

$$Z_n(f, A_{s,k}) = \sum_{x \in Fix\, f^n} \exp A_{s,k}(f^n(x)),$$

where $A_{s,k}(x) = -s \log |f'(x)| + ik \arg(f'(x))$ and s is a fixed complex number and $k \geq 0$ is an integer. The relation is given by

Theorem 1.5.3. *The partition function $Z_n(f, A_{s,k})$ can be expressed as*

$$Z_n(f, A_{s,k}) = \sum_{j=0}^{2} (-1)^j Tr(L_{s,j}^{(k)})^n.$$

Proof of 1.5.3.

This follows from Lemma 2 of [Rue76] taking ϕ equal to $Z_n(f, A_{s,k})$. $\boxed{1.5.3}$

Another important consequence of the fact that the operators $L_{s,i}^{(k)}, i = 0, 1, 2$ are nuclear is that their Fredholm determinants $\det(1 - zL_{s,i}^{(k)})$ are entire functions and satisfy

1.5.4. $$\det(1 - zL_{s,i}^{(k)}) = \exp trace \log(1 - zL_{s,i}^{(k)}).$$

For each $k \geq 0$, the function

1.5.5. $$\zeta_k(z, s) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(f, A_{s,k})$$

is well defined for $|z| < \lim_{n \to +\infty} \frac{1}{n} \log Z_n(f, A_{s,k})$, and it is not difficult to see that $\zeta_k(z, s)$ is analytic for such values of z. Recall that each $\zeta_k(z, s)$ is a particular example of a Ruelle zeta function as introduced by Ruelle in [Rue76]. Using 1.5.4 we can write $\zeta_k(z, s)$ as

$$\zeta_k(z, s) = \prod_{i=0}^{2} (\det(1 - zL_{s,i}^{(k)}))^{(-1)^{i+1}}.$$

Thus, we proved

Lemma 1.5.6. *$\zeta_k(z,s)$ has a meromorphic extension to $\mathbb{C}\times\mathbb{C}$.*

Now we are able to give a short proof that the Selberg zeta function has a meromorphic extension to $\mathbb{C}$, as we promised in the introduction.

Theorem 1.5.7. *The Selberg zeta function defined in* 1.5.1 *has a meromorphic extension to $\mathbb{C}$.*

Proof of 1.5.7.

For $Re(s) > 2$ we have that

$$Z_{\text{Selberg}}(s) = \prod_{k=0}^{\infty}\prod_{\gamma}\left(1 - e^{-i\theta k}e^{(s+k)l(\gamma)}\right)\prod_{\gamma}\left(1 - e^{i\theta k}e^{(s+k)l(\gamma)}\right)$$

$$= \prod_{k=0}^{\infty}\exp\left(-\sum_{\gamma}\sum_{m=1}^{\infty}\frac{1}{m}e^{-m(-i\theta+(s+k)l(\gamma))}\right)$$

$$\exp\left(-\sum_{\gamma}\sum_{m'=1}^{\infty}\frac{1}{m}e^{-m'(i\theta+(s+k)l(\gamma))}\right)$$

$$= \prod_{k=0}^{\infty}\exp\sum_{m=1}^{\infty}\frac{-1}{m}\sum_{p=1}^{\infty}\frac{1}{p}\sum_{\left\{\begin{array}{c} f^p x = x \\ p \text{ least}\end{array}\right\}} e^{-s\log|(f^p)'(x)|-ik\arg((f^p)'(x))m}$$

$$exp\sum_{m'=1}^{\infty}\frac{-1}{m'}\sum_{q=1}^{\infty}\frac{1}{q}\sum_{\left\{\begin{array}{c} f^q x = x \\ q \text{ least}\end{array}\right\}} e^{-s\log|(f^q)'(x)|+ik\arg((f^q)'(x))m'}$$

$$= \prod_{k=0}^{\infty}\exp\sum_{p=1}^{\infty}\frac{-1}{p}\sum_{f^p(x)=x} e^{-s\log|(f^p)'(x)|-ik\arg((f^p)'(x))}$$

$$\exp\sum_{q=1}^{\infty}\frac{-1}{q}\sum_{f^q(x)=x} e^{-s\log|(f^q)'(x)|+ik\arg((f^q)'(x))}$$

$$= \prod_{k=0}^{\infty}\zeta_k^{-1}(1, s+k)\tilde{\zeta}_k^{-1}(1, s+k).$$

where ζ_k and $\tilde{\zeta}_k$ are given by 1.5.5 with partition functions

$$A_{s,k}(x) = \exp(-s\log|f'(x)| + ik\arg(f'(x)))$$

and

$$\tilde{A}_{s,k}(x) = \exp(-s\log|f'(x)| - ik\arg(f'(x)))$$

respectively, we then use Lemma 1.5.6 to finish our claim. $\boxed{1.5.7}$

Chapter 2

Markov partition and deformation spaces of Kleinian groups

In this chapter we define the deformation of a Kleinian group and consider the question of regularity of the Hausdorff dimension of the limit set of geometrically finite purely loxodromic Kleinian groups having the even corners property. For each point in the deformation space of these groups we prove that under real analytic pertubation, the Hausdorff dimension varies real analytic in a neighbourhood of the given point. Based on the work of Maskit, Marden and others we prove that this result is valid for a large class of Kleinian groups.

2.1. Introduction

The purpose of this chapter is to investigate the question of when the Hausdorff dimension of the limit set is an analytic function on the deformation space of a Kleinian group. We restrict ourselves to purely loxodromic, geometrically finite Kleinian groups.

The argument consists of three steps. The first involves showing that, if a (geometrically finite) Kleinian group (without parabolics) has an expanding Markov map, then Hausdorff dimension of the limit set is an analytic function on its deformation space. This is accomplished using the machinery of thermodynamic formalism and ergodic theory following the ideas described in the paper [KKPW89]. The aim is to build an appropriate set up in order to use Bowen's characterization of the Hausdorff dimension of the limit set.

The second step is to develop a condition which implies that a Kleinian group has an expanding Markov mapping. This condition is related to the existence of a fundamental polyhedron with the property that the extension of any geodesic plane containing a side is an integral part of the tesselation. This is the so called *even corners property.*

Theorem 2.1.1. *Let Γ be a purely loxodromic, geometrically finite Kleinian group. Suppose there exists a quasiconformal deformation Γ^0 of Γ which has the even corners property. Then, the Hausdorff dimension of the limit set is an analytic function on $\mathcal{T}(\Gamma)$.*

The third step is to understand when this condition can be realized. We show that it is preserved under Klein combination, and hence needs only to be checked for groups with connected limit set, as it is obviously true for loxodromic cyclic groups. For quasifuchsian groups, one can get similar results from the work of Bowen. Hence, we need to demonstrate that it holds for web and extended quasifuchsian groups.

Theorem 2.1.2. *Let Γ be a purely loxodromic, geometrically finite Kleinian group which is formed from Γ_1 and Γ_2 by Klein combination. Suppose that each Γ_j is quasiconformally conjugate to a group Γ_j^0, where Γ_j^0 has the even corners property. Then, Γ is quasiconformally conjugate to a purely loxodromic, geometrically finite Kleinian group Γ^0 with the even corners property.*

In particular, we have that Hausdorff dimension is an analytic function of the deformation space of a geometrically finite, purely loxodromic function group.

History

Rufus Bowen in [Bow79] was the first one to ask whether the Hausdorff dimension varies continuously over the deformation space of the group. Sullivan posed a harder question when in [Sul83] he asked if the Hausdorff dimension varies real analytically on the deformation space. Later, Ruelle in [Rue76] proved the result for the case of Fuchsian groups without elliptics or parabolics. His proof is based on zeta functions and how their coefficients vary under real analytic perturbation of a parameter.

One can see the problem from a dynamical systems point of view if one uses the work of Sullivan published in [Sul84]. In this paper, he proved that for geometrically finite Kleinian groups without cusps, the Hausdorff dimension of the limit set is equal to the topological entropy of the geodesic flow, restricted to those geodesics whose end points are in the limit set. Katok, Knieper, Pollicott and Weiss studied C^ω (real analytic) perturbations of Anosov and geodesic flows on closed Riemannian manifolds. They showed that the topological entropy varies as smoothly as the perturbations. Therefore, one could try to extend their results to the case of Axiom A flows and use Sullivan's characterization to prove that the Hausdorff dimension varies real analytically under C^ω perturbations. We based our methods on techniques which are rooted in the original work of Bowen. Our methods give a geometrical approach to the problem, and also we construct a "canonical" Markov map associated to the group. Moreover, using this geometrical approach we produced important results related to Kleinian group theory aspects of the problem.

More recently, the result was proved to be false for infinitely generated Fuchsian groups, see [AZ94].

2.2. Markov partitions

In this section, we generalize a construction originally given by R. Bowen and C. Series [BS79] in their work on Markov partitions for Fuchsian groups. They show, among other results, the existence of an expanding Markov map associated to a finitely generated Fuchsian group acting on the Poincaré disc $\mathbb{H}^2$. They consider a finitely generated Fuchsian group Φ having a fundamental polygon D in $\mathbb{H}^2$ with the *even corners property*, that is, $\Gamma(\partial D)$ is a union of lines. Note that the fundamental group of any compact surface of genus $g \geq 2$ has a Fuchsian realization having a fundamental polygon with this property [BS79].

We wish to generalize this construction to Kleinian groups acting on $\mathbb{H}^3$. We begin with a few definitions. By a *polyhedron* in $\mathbb{H}^3$, we mean the intersection of a locally finite collection of closed half spaces; in particular, polyhedra are convex. A *fundamental polyhedron* P for a Kleinian group Γ is a polyhedron in $\mathbb{H}^3$ so that every point of $\mathbb{H}^3$ is a translate of a point of P, the interior $\mathrm{int}(P)$ of P is disjoint from all its translates, and the sides of P are paired by elements of Γ. By this last condition, we mean that, for every side s of P, there is a side s' of P and an element $\gamma_s \in \Gamma \setminus \{id\}$ with $\gamma_s(s) = s'$. We call γ_s a *face pairing* transformation. An example of a fundamental polyhedron for Γ is the *Dirichlet polyhedron* $D_0(\Gamma)$ centered at a point $0 \in \mathbb{H}^3$, given by

$$D_0(\Gamma) = \{x \in \mathbb{H}^3 \mid \mathrm{d}(0,x) \leq \mathrm{d}(\gamma(0),x) \text{ for all } \gamma \in \Gamma \setminus \{id\}\}.$$

We may also consider the action of Γ on $\overline{\mathbb{C}}$. The action of Γ partitions $\overline{\mathbb{C}}$ into two sets. The *ordinary set* $\Omega(\Gamma)$ is the largest open subset of $\overline{\mathbb{C}}$ on which

Γ acts properly discontinuously. The *limit set* $\Lambda(\Gamma)$ is the smallest non-empty closed subset of $\overline{\mathbb{C}}$ which is invariant under the action of Γ, and is the home of much of the interesting dynamical behavior of Γ.

From now on we assume that $\Omega(\Gamma) \neq \varnothing$.

A Kleinian group is *geometrically finite* if there exists a finite sided fundamental polyhedron for its action on $\mathbb{H}^3$. It is a consequence of the Poincaré Polyhedron Theorem [Mas71a] that a geometrically finite Kleinian group is generated by its face pairing transformations; in particular, geometrically finite Kleinian groups are finitely generated.

In the language of Sullivan, Kleinian groups which are purely loxodromic and geometrically finite are *expanding*, in the sense that, for every $x \in \Lambda(\Gamma)$, there is some $\gamma \in \Gamma$ so that $|\gamma'(x)| \geq 1$.

A Kleinian group Γ acting on $\mathbb{H}^3$ has the *even corners property* if there is a fundamental polyhedron P for Γ so that

$$\Gamma(\partial P) = \bigcup_{\gamma \in \Gamma} \gamma(\partial P)$$

is a union of geodesic planes. We sometimes refer to a fundamental polyhedron P satisfying this condition as an *even cornered* fundamental polyhedron for Γ.

Let Γ be a purely loxodromic, geometrically finite Kleinian group which satisfies the even corners property. Recall that a Kleinian group Γ is *purely loxodromic* if every non-trivial element is loxodromic. We will show that we can define an expanding Markov map f on $\Lambda(\Gamma)$ for such Γ. This allows us to study the action of Γ on $\Lambda(\Gamma)$, which can be quite complicated, via a single map f which is significantly easier to analyse.

A *Markov partition* for the action of Γ on $\overline{\mathbb{C}}$ is a finite collection $\mathcal{P} = \{V_i\}_{i=1}^m$ of non-empty closed subsets of $\overline{\mathbb{C}}$ and a map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

(a) $\bigcup_{i=1}^m V_i = \overline{\mathbb{C}}$,

(b) for $i \neq j$, the interior of V_i is disjoint from the interior of V_j, and

(c) the image under f of any element in the partition is a union of elements of the partition; that is, $f(V_i) = V_{j_1} \cup \ldots \cup V_{j_n}$.

We say that such an f has the Markov property with respect to the partition $\mathcal{P}$.

Given an even cornered fundamental polyhedron $\mathcal{R}$ for a Kleinian group Γ, we use the planes in $\Gamma(\partial\mathcal{R})$ to construct a partition of $\overline{\mathbb{C}}$, and use a generating set for Γ which comes from $\mathcal{R}$ to construct the map f.

Let $\mathcal{R}$ be a fundamental polyhedron for Γ, and set

$$\Gamma_{\mathcal{R}} = \{\gamma \in \Gamma \setminus \{id\} : \gamma(\mathcal{R}) \cap \mathcal{R} \text{ is a face of } \mathcal{R}\}.$$

This is a finite subset of Γ and is symmetric, in the sense that $\Gamma_{\mathcal{R}}$ is closed under inverses. Since $\Gamma_{\mathcal{R}}$ consists of the face pairing transformations for Γ, it is a generating set for Γ (though perhaps not minimal).

We use the elements in $\Gamma_{\mathcal{R}}$ to label the faces of $\mathcal{R}$ as follows. Use the label e on a face of $\mathcal{R}$ if the element $e \in \Gamma_{\mathcal{R}}$ pairs it to some other face. Write e inside $\mathcal{R}$ and $\bar{e}$ on the outside, where $\bar{e}$ denotes $e^{-1} \in \Gamma_{\mathcal{R}}$. Let $H(e)$ be the half-space in $\mathbb{B}^3$ whose intersection with $\mathcal{R}$ is the face whose exterior label is e. It is clear that $\{H(e_i) : e_i \in \Gamma_{\mathcal{R}}\}$ and $\mathcal{R}$ have union $\mathbb{B}^3$. Since $H(e_i)$ contains a face of $\mathcal{R}$, we could use the associated face pairing to define a map f that sends a point x in $H(e_i)$ to $e_i^{-1}x$ if i is the least element of $\{j : x \in H(e_j)\}$ and $f(x) = x$ if $x \in \mathcal{R}$.

Consider now the set of all planes in $\Gamma(\partial\mathcal{R})$ that intersect $\mathcal{R}$ in either a face, an edge or a vertex. The intersection of each plane in this collection with $\overline{\mathbb{C}}$ is a circle, and the closures of the components of the complement of the union of these circles covers $\overline{\mathbb{C}}$, giving us a measurable partition $\mathcal{P} = \{R_i\}$ of $\overline{\mathbb{C}}$.

Since the group Γ has no parabolics, this partition is finite. It is immediate to verify that the map f defined in the last paragraph can be extended to $\overline{\mathbb{C}}$ in the same way it was defined. This fact together with the hypothesis that $\mathcal{R}$ is even cornered allowed us to prove the following.

Proposition 2.2.1 ([Roc]). *The map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ has the Markov property with respect to the partition $\mathcal{P}$.*

We will usually restrict our attention to a Markov partition on $\Lambda(\Gamma)$, which we obtain by intersecting each element of the partition $\mathcal{P}$ with $\Lambda(\Gamma)$ and then restricting the action of f. Restricting f caused no difficulty; since f is defined in terms of elements of Γ, it preserves $\Lambda(\Gamma)$.

One of the basic objects in thermodynamic formalism is the notion of a shift of finite type, defined as follows. Given $k > 1$, let A be an aperiodic $k \times k$ matrix whose entries are either 0 or 1; recall that A is *aperiodic* if all the entries of A^n are positive for some $n \geq 1$.

Consider the space Σ_A of all sequences in the alphabet $\{1,\dots,k\}$ such that the letter i is followed by the letter j if and only if $A_{ij}=1$. In other words,

$$\Sigma_A = \left\{ x=(x_n) \in \prod_{n=0}^{\infty}\{1,\dots,k\} : A_{x_n x_{n+1}} = 1 \right\}.$$

Give $\{1,\dots,k\}$ the discrete topology, so that Σ_A with the product topology is a compact space. The (one sided) *shift of finite type* $\sigma=\sigma_A:\Sigma_A\to\Sigma_A$ is defined by $(\sigma x)_i = x_{i+1}$. For each θ in the interval $(0,1)$, there exists a metric $\rho=\rho_\theta$ on Σ_A defined by $\rho(x,y)=\theta^N$, where $N=\sup\{n : x_i=y_i \text{ for } 0\le i\le n\}$. Furthermore, the aperiodicity of A implies that σ is mixing, that is, given any two open sets $U,V\subset\Sigma_A$, there is $n\ge 0$ such that $\sigma^n(U)\cap V\neq\varnothing$.

Following the standard procedures of symbolic dynamics, we introduce a subshift of finite type Σ_f having as alphabet the elements in the partition $\mathcal{P}$. More explicitly, let $\mathcal{P}=\{R_i\}_{i=1}^n$ and set

$$\Sigma_f = \left\{ r=(r_k) \in \prod_{k=0}^{\infty}\{1,\dots,n\} : f(R_{r_k}) \supseteq R_{r_{k+1}} \right\}$$

and define the shift map $\sigma:\Sigma_f\to\Sigma_f$ by $(\sigma r)_k = r_{k+1}$.

A finite sequence $r_0\dots r_n$ is called *admissible* if $f(R_{r_k})\supseteq R_{r_{k+1}}$ for $0\le k\le n-1$. If $r_0\dots r_n$ is admissible, define

$$R(r_0\dots r_n) = \bigcap_{i=0}^{n} f^{-i}(R_{r_i}).$$

Lemma 2.2.2 ([Roc]). *diam*$(R(r_0\dots r_n)) \overset{n\to\infty}{\longrightarrow} 0.$

Notation: diam$(R(r_0\dots r_n))$ stands for the Euclidean diameter of the set $R(r_0\dots r_n)$.

The combination of Lemma 2.2.2 with the even corner condition on $\mathcal{R}$ allows us to prove that f is eventually expanding.

Proposition 2.2.3 ([Roc]). *There exists a $\beta>0$ and N so that $|(f^N)'(x)|\ge\beta>1$ for all $x\in R(r_0,\dots,r_N)$.*

We will later need to consider open neighbourhoods of each $R_i\in\mathcal{P}$, which are defined as follows. Let S_i be a small neighbourhood in $\overline{\mathbb{C}}$ of R_i. We choose S_i small enough in order to guarantee that we still have

2.2.4. $$|(f^N)'(x)|\ge\beta'>1 \text{ for all } x\in S(r_0\dots r_N),$$

where $S(r_0\dots r_N)=\cap_{i=0}^N f^{-i}(S_i)$.

2.3. Deformation spaces and expanding maps

In this section, we give a brief description of the deformation space $\mathcal{T}(\Gamma)$ of a finitely generated Kleinian group Γ, and review some basic facts which we will make use of later. Much of the background material relating to this section can be found in Bers [Ber70] and Maskit [Mas71b].

Let $\mathcal{M}$ be the complex Banach space of all complex valued measurable functions on $\overline{\mathbb{C}}$, with the norm $\|\mu\| = \text{ ess sup}(|\mu|)$. Given a Kleinian group Γ, set

$$\mathcal{M}(\Gamma) = \{\mu \in \mathcal{M} \mid \mu(\gamma(z)) \cdot \overline{\gamma'(z)} = \mu(z) \cdot \gamma'(z) \text{ for all } \gamma \in \Gamma, \text{ a.e.} z \in \Omega(\Gamma)\},$$

and note that $\mathcal{M}(\Gamma)$ is again a complex Banach space, as it is a closed subspace of $\mathcal{M}$.

For each $\mu \in \mathcal{M}(\Gamma)$ with $\|\mu\| < 1$, let $\omega = \omega^\mu$ be the unique solution of the Beltrami equation

2.3.1.
$$\omega_{\overline{z}} = \mu \cdot \omega_z$$

fixing 0, 1, and ∞; the Theorem demonstrating the existence of such a solution is often referred to as the *measurable Riemann mapping theorem*, and a proof can be found in [AB60]. Note that the condition on the elements of $\mathcal{M}(\Gamma)$ is exactly the condition needed to assert that $\omega \circ \gamma \circ \omega^{-1}$ is again an element of $\mathrm{PSL}_2(\mathbb{C})$.

Hence, the group $\Gamma^\mu = \omega^\mu \Gamma(\omega^\mu)^{-1}$ is again a Kleinian group, which we call a *deformation* or *quasiconformal deformation* of Γ. The deformation space of Γ will be the set of equivalence classes of a certain equivalence relation on $\mathcal{M}(\Gamma)$.

Define an element $\sigma \in \mathcal{M}(\Gamma)$ to be *trivial* if $\omega^\sigma \circ \gamma \circ (\omega^\sigma)^{-1} = \gamma$ for all $\gamma \in \Gamma$. Note that, if σ is trivial, then ω^σ is the identity when restricted to $\Lambda(\Gamma)$, as ω^σ fixes the fixed points of all elements of Γ and these fixed points are dense in $\Lambda(\Gamma)$.

Say that two elements μ and τ of $\mathcal{M}(\Gamma)$ are *equivalent* if there is a trivial $\sigma \in \mathcal{M}(\Gamma)$ so that

$$\omega^\mu = \omega^\tau \circ \omega^\sigma.$$

In particular, if μ and τ are equivalent, then $\omega^\mu|_{\Lambda(\Gamma)} = \omega^\tau|_{\Lambda(\Gamma)}$.

Denote the equivalence class of $\mu \in \mathcal{M}(\Gamma)$ by $[\mu]$. The quotient of $\mathcal{M}(\Gamma)$ by this equivalence relation is the *deformation space* $\mathcal{T}(\Gamma)$ of Γ. Using the complex structure which comes from the canonical projection $\mathcal{M}(\Gamma) \to \mathcal{T}(\Gamma)$, $\mathcal{T}(\Gamma)$ is a complex manifold (see, for example, [Ber70]).

It is also possible to see this complex structure by embedding $\mathcal{T}(\Gamma)$ as an open set in a complex manifold. Given $\mu \in \mathcal{M}(\Gamma)$, consider the discrete, faithful representation $\rho^\mu : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ given by $\rho^\mu(\gamma) = \omega^\mu \circ \gamma \circ (\omega^\mu)^{-1}$. Note that the representation depends only on the equivalence class of μ, as trivial elements of $\mathcal{M}(\Gamma)$ induce the identity representation. With the choice of a generating set $\gamma_1, \ldots, \gamma_k$ for Γ, this gives a holomorphic embedding of $\mathcal{T}(\Gamma)$ as a complex submanifold of $X = PSL_2(\mathbb{C}))^k$, given by

$$[\mu] \mapsto (\rho^\mu(\gamma_1), \ldots, \rho^\mu(\gamma_k)).$$

We now have the necessary material to construct Markov maps associated to groups in the deformation space $\mathcal{T}(\Gamma)$ of Γ. We use ω^μ to transport the Markov partition for Γ to a Markov partition for Γ^μ. Recall that we determined a Markov partition for the whole of the sphere at infinity. We can get a partition of the limit set $\Lambda(\Gamma)$ of Γ with the same properties by intersecting each element of $\mathcal{P}$ with $\Lambda(\Gamma)$ and restricting the action of f. By a slight abuse of language, we denote this partition by $\mathcal{P}$ and its elements by $R_1, \ldots, R_n$.

Given $[\mu] \in \mathcal{T}(\Gamma)$, define $f_\mu = \omega^\mu \circ f \circ (\omega^\mu)^{-1}$ and $\mathcal{P}^\mu = \{R_i^\mu = \omega^\mu(R_i) : R_i \in \mathcal{P}\}$; this gives a Markov partition for the action of Γ^μ on its limit set. Note that, if we choose τ equivalent to μ, then ω^τ and ω^μ are equal when restricted to $\Lambda(\Gamma)$, and so $f_\mu = f_\tau$ and $\mathcal{P}^\mu = \mathcal{P}^\tau$. Therefore, the shifts on Σ_{f_μ} and Σ_{f_τ} are topologically the same, and so either one of them is an appropriate shift to describe the dynamics occurring on $\Lambda(\Gamma^\mu)$.

So, choose a representative μ for $[\mu] \in \mathcal{T}(\Gamma)$. Define $S_i^\mu = \omega^\mu(S_i)$ of each R_i^μ, where the S_i are the neighborhoods for R_i chosen above.

Proposition 2.3.2. *The map f_μ is (eventually) expanding.*

Proof of 2.3.2. Define

$$R^\mu(r_0 \ldots r_n) = \omega^\mu(R(r_0 \ldots r_n))$$

and

$$S^\mu(r_0 \ldots r_n) = \omega^\mu(S(r_0 \ldots r_n)).$$

Let $T : S^\mu(r_0 \cdots r_n) \to S^\mu(r_0 \cdots r_n)$ be given by

2.3.3. $$T = \omega^\mu \cdot (f^{-n}|_{S(r_0 \cdots r_n)}) \cdot (\omega^\mu)^{-1}.$$

From this point on we follow the argument used to prove Lemma 3 of Bowen [Bow79]. We can assume, without loss of generality, that $\infty \in \Omega(\Gamma)$. Let γ_n be a closed curve bounding a region which contains R_n^μ and interior to S_n^μ. Given $y \in R_n^\mu$ we then have

$$|T'(y)| = \frac{1}{2\pi}\left|\int_{\gamma_n} \frac{T(\zeta)d\zeta}{(\zeta - y)^2}\right| \leq C \operatorname{diam}(T(S_n^\mu)),$$

for some constant C. Using 2.3.3 we have that $T(S_n^\mu) = \omega^\mu(S(r_0 \ldots r_n))$. Since ω^μ is a homeomorphism and $\operatorname{diam}(S(r_0 \ldots r_n)) \xrightarrow{n\to\infty} 0$, then for some N we have

$$|(f_\mu^N)'(x)| \geq (\sup\{|T'(y)| : y \in R_N\})^{-1} > 1.$$

$\boxed{2.3.2}$

2.4. The Hausdorff dimension of the limit set

The purpose of this section is to give a proof of Theorem 2.1.1. The proof is a re-interpretation of the results obtained by Katok, Knieper, Pollicott and Weiss on the analyticity of the topological entropy for Anosov flows, see [KKPW89], to the present case together with a characterization of the Hausdorff dimension of the limit set of groups in $\mathcal{T}(\Gamma)$ which is due to Bowen, see [Bow79].

We begin with some definitions, using the notation of the previous section.

Let $C(\Sigma_A)$ be the space of complex valued continuous functions on Σ_A. Given $f \in C(\Sigma_A)$, define

$$\operatorname{var}_n(f) = \sup\{|f(x) - f(y)| : x_i = y_i, 0 \leq i \leq n\}$$

and

$$|f|_\theta = \sup_{n\geq 0} \frac{\operatorname{var}_n(f)}{\theta^n}.$$

Set $\mathcal{F}_\theta = \{f \in C(\Sigma_A) : |f|_\theta < \infty\}$. For $f \in \mathcal{F}_\theta$, define

$$\|f\|_\theta = \|f\|_\infty + |f|_\theta.$$

Then, $\|\cdot\|_\theta$ is a norm on $\mathcal{F}_\theta$ and with respect to this norm $\mathcal{F}_\theta$ is a Banach space.

Definition 2.4.1 (Ruelle operator). For a given function $f \in C(\Sigma_A)$, define the Ruelle operator $\mathcal{L}_f : C(\Sigma_A) \to C(\Sigma_A)$ by

$$(\mathcal{L}_f\, g)(x) = \sum_{\sigma(y)=x} e^{f(y)} g(y).$$

Let $\mathcal{L}_f^*$ denote the adjoint of $\mathcal{L}_f$.

Theorem 2.4.2 (Ruelle-Perron-Frobenius see [PP90]). *For each real valued function $f \in \mathcal{F}_\theta$, there is a simple maximal positive eigenvalue λ_f of $\mathcal{L}_f$ with a corresponding strictly positive eigenfunction $h \in \mathcal{F}_\theta$. The remainder of the spectrum of $\mathcal{L}_f : \mathcal{F}_\theta \to \mathcal{F}_\theta$ (excluding λ_f) is contained in a disc of radius smaller than λ_f centered at the origin. There is a unique Borel probability measure μ such that $\mathcal{L}_f^* \mu = \lambda_f \mu$ (that is to say, $\int \mathcal{L}_f v \, d\mu = \lambda_f \int v \, d\mu$) for all $v \in C(\Sigma_A)$. If the eigenvector h satisfies $\int h \, d\mu = 1$, then*

$$\frac{1}{\lambda_f^n} \mathcal{L}_f^n v \to h \int v \, d\mu$$

uniformly for all $v \in C(\Sigma_A)$.

Given a real valued $u \in \mathcal{F}_\theta$, define $P(u) = \log \lambda_u$ to be the *pressure function.*

Theorem 2.4.3 (Variational principle see [Bow75]). *For any real valued $u \in \mathcal{F}_\theta$,*

2.4.4.

$$P(u) = \sup\{h(\mu, \sigma) + \int u \, d\mu : \mu \text{ is a } \sigma\text{-invariant probability measure}\},$$

where $h(\mu, \sigma)$ is the measure theoretic entropy of σ with respect to μ. There exists a unique probability measure μ such that $P(u) = h(\mu, \sigma) + \int u \, d\mu$. This measure is called the equilibrium state.

We now analyse the dependence of s_μ with respect to parameters as $[\mu]$ varies in $\mathcal{T}(\Gamma)$. More specifically, we want to study the dependence of s_μ when given $[\mu] \in \mathcal{T}(\Gamma)$ we choose $\varepsilon > 0$ small enough so that we can define a real analytic perturbation $[\mu]_t$, $t \in (-\varepsilon, \varepsilon)$, of $[\mu] = [\mu]_0$. Let $\delta_{\mathrm{H}}(X)$ be the Hausdorff dimension of $X \subset \overline{\mathbb{C}}$.

Proposition 2.4.5. *(see* [Rue82]*) Define*

$$\phi_\mu(x) = -\log |f'_\mu(x)|.$$

Then,

$$\delta_{H}(\Lambda(\Gamma^\mu)) = s_\mu,$$

where s_μ is the unique real number such that $P(s_\mu \phi_\mu) = 0$.

Consider the function

$$\begin{aligned} \delta : \mathcal{T}(\Gamma) &\to \mathbb{R} \\ [\mu] &\mapsto \delta_H(\Lambda(\Gamma^\mu)). \end{aligned}$$

Theorem 2.4.6. *Let Γ be a purely loxodromic, geometrically finite Kleinian group which has the even corners property. Suppose there exists a quasiconformal deformation Γ^0 of Γ which has the even corners property. Then, the function δ_H is a real analytic function on $\mathcal{T}(\Gamma)$.*

Proof of 2.4.6. Let $[\mu]$ be a point in $\mathcal{T}(\Gamma)$. Recall that we get a Kleinian group Γ^μ if we conjugate all the elements in Γ by the solution of the Beltrami equation given in 2.3.1. In the context that follows, we consider these groups as being subgroups of $PSL_2(\mathbb{C})$

We can use the results described in the last section to define a decomposition $\mathcal{P}^\mu = \{R_i^\mu\}$ of $\Lambda(\Gamma^\mu)$ and a Markov map f_μ. We use the partition $\mathcal{P}^\mu$ and f_μ to construct a symbolic coding for the points in $\Lambda(\Gamma^\mu)$, that is to say we construct the shift of finite type Σ_{f_μ}. We remind the reader that f_μ is an expanding mapping, as shown in Proposition 2.3.2. It is important to mention that the N for eventually expandig in Proposition 2.3.2 is independent of the point taken in the deformation space of the group Γ. This is due to the fact that the symbolic coding is the same for all Γ^μ in (Γ).

Let $\varepsilon > 0$ be sufficiently small so that for $t \in (-\varepsilon, \varepsilon)$, we have a real analytic pertubation μ_t of $\mu = \mu_0$. For each Γ^{μ_t}, with t fixed, consider the function $d_t(s) = 1/\zeta_t(s)$, where

2.4.7.
$$\zeta_t(s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in Fix\, f_{\mu_t}^n} \exp\left(-s \sum_{k=0}^{n-1} \phi_{\mu_t}(f_{\mu_t}^k x) \right).$$

From the results proved by Pollicott in [Pol86], it follows that $d_t(s)$ is analytic for $\Re(s) > s_\mu - \delta$, for some $0 < \delta < s_\mu$, where s_μ is defined in Proposition 2.4.5. There are no zeros for $\Re(s) > s_\mu$ and there is only a single simple zero on $\Re(s) = s_\mu$ and this zero is located at $s = s_\mu$.

We now pass to the proof that $d_t(s)$ is real analytic in t for $\Re(s)$ sufficiently large. The objective is to show that the coefficients of $d_t(s)$, which are functions of the fixed points of $f_{\mu_t}^n$, depend real analytically on $t \in (-\varepsilon, \varepsilon)$ and then extend each of them to a neighbourhood $V \subset \mathbb{C}$ which is independent of the particular coefficient.

Once the groups we are considering are purely loxodomic, we have that $f_{\mu_t}^n$ is a loxodromic transformation γ_{μ_t}. Given a fixed point of $f_{\mu_t}^n$ we can calculate the displacement distance $l(\gamma_{\mu_t})$. We say that γ_{μ_t} is primitive if it is not conjugate in Γ^{μ_t} to a power of any element in Γ^{μ_t} (including its own inverse). We have that

2.4.8.
$$d_t(s) = \prod_{\gamma_{\mu_t}\ i.p.} \left(1 - e^{-sl(\gamma_{\mu_t})}\right),$$

where the product runs over all non-conjugate primitive elements. Given a loxodromic element γ belonging to the unperturbed group Γ^{μ}, we know that for each $t \in (-\varepsilon, \varepsilon)$ there is a unique loxodromic element $\gamma_t \in \Gamma^{\mu_t}$ which corresponds to γ. This is a consequence of the way we construct the deformation space in section 2.3. Consider the function $t \mapsto l_t(\gamma)$, where $l_t(\gamma)$ is the displacement distance of the loxodromic element $\gamma_t \in \Gamma^{\mu_t}$. The proof of the following crucial Lemma is an adaptation to our case of the proof given to Proposition 1.1 of [KKPW89].

Lemma 2.4.9. *Let γ be a fixed element in Γ^{μ}, then the function $t \mapsto l_t(\gamma)$ is real analytic in $(-\varepsilon, \varepsilon)$ and it extends to a holomorphic function in $V \subset \mathbb{C}$, such that V does not depend on the choice of γ.*

Proof of 2.4.9. Once μ_t depends real analytically on t, it follows from the measurable Riemann mapping theorem that the traces of all elements in each Γ^{μ_t} depend real analytically on μ_t. Since $l_t(\gamma)$ can be expressed as a real analytic function of $trace(\gamma_t)$, the first part of the Lemma follows. Still by the measurable Riemann mapping theorem, we have that f_{μ_t} depends real analytically on μ_t because it is defined in terms of the generators of Γ^{μ_t}. Due to the fact that both l_t and f_{μ_t} are real analytic function on $(-\varepsilon, \varepsilon)$, we can extend them to a complex open neighbourhood of $(-\varepsilon, \varepsilon)$. We denote these extensions by l_t and f_{μ_t} respectively.

Let $\mathcal{P} = \{R_1^{\mu}, \ldots, R_n^{\mu}\}$ be the Markov partition for the unperturbed group Γ^{μ}, and f_{μ} the respective expanding Markov map. For each R_i^{μ} choose a neighbourhood $\tilde{R}_i^{\mu}$ such that $int(\tilde{R}_i^{\mu}) \supset R_i^{\mu}$. Let $R_i^{\mu} R_j^{\mu}$ be a pair of admissible symbols, see end of section 2.2. We have arrange that $int(f_{\mu}\tilde{R}_i^{\mu}) \supseteq \tilde{R}_j^{\mu}$. Choose $V_{ij} \subset \mathbb{C}$, such that $V_{ij} \supset (-\varepsilon, \varepsilon)$ and for $t \in V_{ij}$ we still have this configuration, i.e. $int(f_{\mu_t}\tilde{R}_i^{\mu}) \supseteq \tilde{R}_j^{\mu}$ if $t \in V_{ij}$. Define for each $t \in V_{ij}$ the function $p_t : \tilde{R}_j^{\mu} \to \tilde{R}_i^{\mu}$ given by $p_{ij}^t \equiv f_{\mu_t}^{-1}$. We can assume that p_{ij}^t is a contraction, if not we just have to take some iterate of $f_{\mu_t}^{-1}$. If x is a fixed point for f_{μ}^n, $n \geq 1$,then it is the end point of the axis of a loxodromic element in Γ^μ. It is then easy to see that x has a periodic f_μ-expansion with period, say, $r_0 r_1 \cdots r_{n-1}$ such that $f_\mu(R^\mu_{r_i}) \supseteq R^\mu_{r_{i+1(\mathrm{mod}\ n)}}$. It is important to note that the coding of such a point is the same no matter which value of μ we are considering. Define

$$V = \bigcap_{f_\mu(R_i)\supseteq R_j} V_{ij}.$$

Given a periodic sequence $r_0 r_1 \cdots r_{n-1}$ as above we choose $t \in V$ in order to get a composition of contraction maps

$$\tilde{R}^\mu_{r_0} \stackrel{p^t_{r_0 r_{n-1}}}{\longrightarrow} \tilde{R}^\mu_{r_{n-1}} \stackrel{p^t_{r_{n-1} r_{n-2}}}{\longrightarrow} \cdots \tilde{R}^\mu_{r_1} \stackrel{p^t_{r_1 r_0}}{\longrightarrow} \tilde{R}^\mu_{r_0}.$$

We now use the contraction mapping theorem to guarantee the existence of a unique fixed point x_t in $\tilde{R}^\mu_{r_0}$ for the above composition. Therefore we can say that $(f_{\mu_t})^n$ has a fixed point in $\tilde{R}^\mu_{r_0}$. Once $V \supset (-\varepsilon, \varepsilon)$, we apply the implicit function theorem to conclude that the function $V \to \tilde{R}^\mu_{r_0}$ given by $t \mapsto x_t$ is holomorphic.

Recall that for $t \in (-\varepsilon, \varepsilon)$, each $(f^{\mu_t})^n$ is a fractional linear transformation $g_{\mu_t}(x) = \frac{a_t x + b_t}{c_t x + d}$. Given a fixed point x_t of $(f_{\mu_t})^n$, the displacement distance distance $l_t(g^\mu) = l_t(g^{\mu_t})$ of g_{μ_t} is given by

2.4.10.
$$l(g_{\mu_t}) = \frac{1}{|c_t x_t + d_t|^2}.$$

Since $l_t(g^\mu)$ extends to a holomorphic function defined in V, we use (2.4.10) to finish the claim. $\boxed{2.4.9}$

We now return to the proof that $d_t(s)$ has a holomorphic extension to $V \subset \mathbb{C}$, where V is the neighbourhood obtained from Lemma 2.4.9. After some elementary manipulation we have that

$$d_t(s) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\gamma_\mu\, i.p.} e^{-s l_t(\gamma_\mu) n} \right).$$

Observe that from the above the analyticity of $d_t(s)$ in $t \in V$ follows if we prove that the series $\sum_{\gamma_\mu\, i.p.} |e^{-s l_t(\gamma_\mu) n}|$ converges uniformly $t \in V$. This follows because we can find a neighbourhood $V' \subset V$ of $(-\varepsilon, \varepsilon)$ such that for $T \geq 1$

$$\sharp\{\gamma_\mu : l_t(\gamma_\mu) \leq T\} \leq e^{cT},$$

where c is a constant independent of $t \in V'$, then if $\Re(s)$ is sufficiently large we have $d_t(s)$ is holomorphic in t in a neighbourhood V' of $(-\varepsilon, \varepsilon)$.

We summarize the conclusion we obtained so far. We have that for $\Re(s)$ sufficiently large $d_t(s)$ is a bi-analytic function in s and in t. From Pollicott's result, stated above, we also know that for fixed t, $d_t(s)$ has an analytic extension to the larger half plane $\Re(s) > s_\mu - \delta$, and from the last paragraph we saw that $d_t(s)$ is holomorphic in t for $\Re(s)$ large enough. We want to show that $d_t(s)$ is also bi-analytic in the larger half-plane $\Re(s) > s_\mu - \delta$. This follows from the analysis carried out in [KKPW89] which was based on the following theorem.

Theorem 2.4.11 ([Shi89]). *Let $f : (-1,1) \times \Delta_2 \to \mathbb{C}$, where $\Delta_2 = \{z \in \mathbb{C} : |z| < 2\}$, and let $0 < a < 2$ be such that*

- *for every $x \in (-1,1)$, $f(x,\cdot)$ is holomorphic in Δ_2,*
- *for every $z \in \Delta_a$, $f(\cdot,z)$ is C^ω on $(-1,1)$.*

Then for $r < 2$, there exists an open set $U \subset \mathbb{C}$ such that $U \cap \mathbb{R} = (-1,1)$, and a holomorphic function $\tilde{f}$ on $U \times \Delta_r$ such that $\tilde{f}_{(-1,1)\times\Delta_r} = f$.

Once we now know that $d_t(s)$ is bi-analytic in (s,t) and that its zeros s_{μ_t} are simple, we then prove the result claimed using the Implicit function Theorem. $\boxed{2.4.6}$

2.5. Even corners and Klein combination

Up to this point, we have been concerned with showing that the even corners property is sufficient to guarantee that the Hausdorff dimension of the limit set is an analytic function on the deformation space of a purely loxodromic, geometrically finite Kleinian group (Theorem 2.4.6). In this section, we begin the discussion of which of these groups have the even corners property.

More specifically, we show that the even corners property is preserved by Klein combination, in the following weak sense. If Γ is formed from Γ_1 and Γ_2 by Klein combination, and if Γ_1 and Γ_2 are quasiconformally conjugate to groups which have the even corners property, then Γ is quasiconformally conjugate to a group with the even corners property. This, when combined with an inductive argument due originally to Abikoff and Maskit, proves that it suffices to demonstrate that purely loxodromic, geometrically finite Kleinian groups with connected limit set have the even corners property.

We begin with a description of the operation of Klein combination. Let Γ_1 and Γ_2 be Kleinian groups and let $S_j = \Omega(\Gamma_j)/\Gamma_j$ be the (possibly disconnected) quotient surface associated to Γ_j. Klein combination is the formalization, at the level of Kleinian groups, of the following two operations. First, take a small disc B_j on S_j, cut it out to get a surface whose boundary is a circle, and glue $S_1 - B_1$ to $S_2 - B_2$ along the boundary circles. Second, take two small discs B_1 and B_2 on S_2, cut them out to again get a surface whose boundary is two circles, and glue the two boundary components of $S_2 - (B_1 \cup B_2)$ together.

Recall that a *fundamental domain* for the action of a Kleinian group Γ on its ordinary set $\Omega(\Gamma)$ is an open set $D \subset \Omega(\Gamma)$ so that distinct translates of D are disjoint and every point of $\Omega(\Gamma)$ is equivalent under the action of Γ to a point of $\overline{D}$. We may always assume, without loss of generality, that the intersection of D with each component of $\Omega(\Gamma)$ is connected; further, in the case that Γ is finitely generated, we may assume that the boundary of D is a finite collection of analytic arcs.

Theorem 2.5.1. *(see* [Mas88]*) Let Γ_1 and Γ_2 be Kleinian groups. Suppose there exist fundamental domains D_j for the action of Γ_j on $\Omega(\Gamma_j)$ so that D_1 contains the complement of D_2 in $\overline{\mathbb{C}}$ and D_2 contains the complement of D_1 in $\overline{\mathbb{C}}$. Then, the group $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is a Kleinian group isomorphic to $\Gamma_1 * \Gamma_2$, and $D = D_1 \cap D_2$ is a fundamental domain for the action of Γ on $\Omega(\Gamma)$.*

We remark that the operation of Klein combination preserves the properties of a Kleinian group being purely loxodromic and geometrically finite. That is, if Γ is formed from Γ_1 and Γ_2 by Klein combination, then Γ is purely loxodromic and geometrically finite if and only if both Γ_1 and Γ_2 are purely loxodromic and geometrically finite.

Theorem 2.5.2. *Let Γ be a purely loxodromic, geometrically finite Kleinian group which is formed from Γ_1 and Γ_2 by Klein combination. Suppose that each Γ_j is quasiconformally conjugate to a Kleinian group Γ_j^0, where Γ_j^0 has the even corners property. Then, Γ is quasiconformally conjugate to a purely loxodromic, geometrically finite Kleinian group Γ^0 with the even corners property.*

Proof of 2.5.2. Suppose that Γ is formed from Γ_1 and Γ_2 from Klein combination, where the Γ_j are quasiconformally conjugate to Kleinian groups Γ_j^0 which have the even corners property. Let P_j' be an even cornered fundamental polyhedron in hyperbolic 3-space for Γ_j^0, and let D_j' be the interior of the boundary at infinity of P'_j. Note that D'_j is then a fundamental domain for the action of Γ^0_j on $\Omega(\Gamma^0_j)$.

Conjugating by Möbius transformations, we may assume that the disc $\{|z| > 1\}$ lies in D'_1 and that the disc $\{|z-10| > 1\}$ lies in D'_2. Then, the complement of D'_1 lies in $\{|z| < 1\}$, which is contained in D'_2; similarly, the complement of D'_2 lies in $\{|z-10| < 1\}$, which is contained in D'_1.

The Klein combination theorem yields that $\Gamma^0 = \langle \Gamma^0_1, \Gamma^0_2 \rangle$ is a purely loxodromic, geometrically finite Kleinian group. which is a quasiconformal deformation of Γ, as proved in the Appendix. Poincaré's Polyhedron Theorem [Mas71a] yields that $P' = P'_1 \cap P'_2$ is a fundamental polyhedron in $\mathbb{H}^3$ for the action of Γ^0, and P' is even cornered by construction. $\boxed{2.5.2}$

It is a result of Abikoff and Maskit [AM77] that, if Γ is a purely loxodromic, geometrically finite Kleinian group, then there exist finitely many subgroups $\Gamma_1, \ldots, \Gamma_p$ of Γ, where each Γ_j is either loxodromic cyclic, quasifuchsian, extended quasifuchsian, or web, so that Γ is constructed from the Γ_j by finitely many applications of the Klein combination theorem.

Each loxodromic cyclic Kleinian group $\langle\gamma\rangle$ has the even corners property; let H be any hyperplane in $\mathbb{H}^3$ which is orthogonal to the axis of γ and consider the polyhedron whose sides are H and $\gamma(H)$. As noted by Bowen [Bow79], every quasifuchsian group is quasiconformally conjugate to a Fuchsian group which has the even corners property.

One immediate consequence of this is the following Corollary.

Corollary 2.5.3. *Let Γ be a purely loxodromic, geometrically finite function group. Then, Γ is quasiconformally conjugate to a Kleinian group with an even cornered fundamental polyhedron. In particular, δ_H is an analytic function on $\mathcal{T}(\Gamma)$.*

Proof of 2.5.3. Every purely loxodromic, geometrically finite function group can be constructed from finitely many loxodromic cyclic and quasifuchsian groups using the Klein combination theorem. Apply Theorem 2.5.2 finitely many times to see that Γ is quasiconformally conjugate to a group with the even corners property, and then apply Theorem 2.4.6. $\boxed{2.5.3}$

Conjecture 2.5.4. Every extended quasifuchsian group is quasiconformally conjugate to an even-cornered group.

Conjecture 2.5.5. Every purely loxodromic, geometrically finite web group Γ is quasiconformally conjugate to an even-cornered group.

One possible approach to Conjecture 2.5.5 is to look at the 'most regular' fundamental polyhedron for each point in $\mathcal{T}(\Gamma)$, and to try and use a variational approach to find the most regular over the whole deformation space.

2.6. Examples

The purpose of this last section is to describe an example of a group for which Theorem 2.4.6 holds.

Example 2.6.1 (A web group). This is an example constructed by Maskit which is described in [Mas88].

Consider the set of six circles $C_1, \ldots, C_6$ given by $|z - \sqrt{2}e^{2\pi i m/6}| = 1$, $m = 1, \ldots, 6$. It is not hard to see that any C_i intersects only two others C_j,C_k, $i \neq j \neq k$, at 90 degrees and that they all intersect the unit circle orthogonally. We get six more circles $C'_1, \ldots, C'_6$ applying the map $z \mapsto (\sqrt{2}+\sqrt{3})e^{\pi i/6}$ to each of the six initial ones. The result is shown in picture below. Observe that either we have C_i intersecting C'_l orthogonally or they are disjoint. Let r_m denote reflection with respect to C_m and let r'_m denote reflection in C'_m. The hyperbolic polyhedron $\mathcal{R}$ in hyperbolic 3-space formed by the planes whose intersection with the Riemann sphere are the circles we described is a fundamental polyhedron for the group Γ generated by $r_1, \ldots, r_6, r'_1, \ldots, r'_6$. Note that since $\mathcal{R}$ has no vertices on the sphere at infinity, the group G contains no parabolics. We get a Kleinian group Γ taking the subgroup of index two formed by the elements in G which preserve orientation. The components of Γ are the translates of the unit disc and of the disc $|z| \geq \sqrt{2} + \sqrt{3} \cup \{\infty\}$. Therefore, the limit set $\Lambda(\Gamma)$ of Γ is a circle packing in which all circles are disjoint.

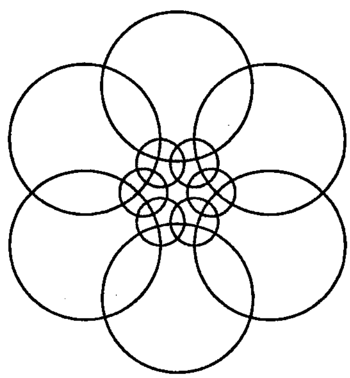

Figure 2.1. A web group. Each circle span a hyperbolic plane in $\mathbb{H}^3$ to form a twelve sided polyhedron with all dihedral angles equal to $\pi/2$

Chapter 3

Markov partitions and automatic structures

In this chapter we deal with Fuchsian groups, symbolic coding of points in their limit set and the word problem. Our aim is to show that this coding allows us to construct an automaton which accepts shortest representatives of elements in the group, and moreover the natural map from the language to the group is a bijection. Following that, we show that these groups have the fellow traveler property, more explicit we show that two fellow travelers can be at most distance one apart. This is done using a very geometrical approach. We therefore provide a new proof that the groups we consider are automatic. The automatic structure we provide has a very strong connection with the ergodicity of the geodesic flow. All the results we prove are based on the works of Series and Birman-Series.

3.1. Introduction

Max Dehn was probably the first one to use a geometrical approach to the solution of the word problem for Fuchsian groups. We can find geometrical motivations of the same sort in the work of several other mathematicians not only when considering the solution of the word problem, but also in the construction of a symbolic coding for geodesics in surfaces of constant negative curvature. This is particularly present in the works of Koebe and Morse.

More recently, a large class of finitely generated groups which has a solvable word problem was determined. The groups in this class are called *automatic groups*. Examples of groups in this class, to quote a few, are the free groups, free abelian groups and Gromov's word hyperbolic groups. The definition of an automatic group was invented by W. Thurston as a reinterpretation of the results of J. Cannon described in [Can84]. For the purpose of this introduction, it is enough to think of an automatic group as an object that has a finite recursive structure which we will call an *automatic structure*. The reference for the theory of automatic groups is David Epstein et al's [ECH+92].

Although it was hinted that the methods created in the spirit of Dehn's ideas and the theory on automatic groups are related, this connection was never fully explored and explained. It is our purpose to describe this relation. More specifically, we will construct an automatic structure for a large class of Fuchsian groups based on the results of C. Series' [Ser91], [Ser81], [Ser86] works and on the Birman-Series paper [BS87].

We will be able to explicitly describe this automatic structure and to determine important aspects of its complexity. One more interesting aspect of this automatic structure is its relation with the symbolic dynamics of the geodesic flow and the fact that this is an ergodic flow. One can consider these automatic structures as finite directed graphs with a precise specification of the labelling of its edges. Because our construction is based on the symbolic dynamics of a certain subshift of finite type which models the geodesic flow, this graph is, apart from one vertex, strongly connected. This property was for instance required in the work of Pollicott and Sharp [PS94] on their study of meromorphic extensions of Poincaré series associated to the groups we consider.

3.2. Preliminaries

We realize hyperbolic space in two dimensions using the Poincaré disc model $\mathbb{D}$ which is the open unit disc in $\mathbb{R}^2$ with the (hyperbolic) metric

3.2.1.
$$ds = \frac{2|dz|}{1-|z|^2}.$$

Geodesics in $\mathbb{D}$ are Euclidean circles orthogonal to $\partial\mathbb{D} = S^1$. The group of conformal maps of $\mathbb{D}$ onto itself is denoted by $\mathrm{Aut}(\mathbb{D})$, and it consists of all fractional linear transformations of the form

$$g(z) = \frac{az+\overline{c}}{cz+\overline{a}}, \quad |a|^2 - |c|^2 = 1.$$

The elements in the orientation preserving isometries group, with respect to the metric given in 3.2.1, are precisely the maps in $\mathrm{Aut}(\mathbb{D})$. A Fuchsian group is a discrete subgroup of $\mathrm{Aut}(\mathbb{D})$. Let Γ be a Fuchsian group. This group acts on the unit disc isometrically and discontinuously, therefore the orbit of any point z accumulates on $\partial\mathbb{D}$. We then define the limit set Λ of Γ as the closure on $\partial\mathbb{D}$ of the set of accumulation points of $\Gamma \cdot z$.

Let $\mathcal{R}$ be a convex geodesic polygon with a finite number of sides which is a fundamental domain for the action of a Fuchsian group Γ. For each side of $\mathcal{R}$ there exists a unique element in Γ which pairs this side to another side of $\mathcal{R}$, we call this element a side pairing transformation. Let $\Gamma_{\mathcal{R}}$ be the set of side pairing transformations. Poincaré's theorem gives us a presentation $\langle \Gamma_{\mathcal{R}} | R_{\mathcal{R}} \rangle$ for Γ with $\Gamma_{\mathcal{R}}$ as a symmetric system of generators and $R_{\mathcal{R}}$ as the set of relations we get from the vertex cycles.

We label each side of $\mathcal{R}$ using the elements in $\Gamma_{\mathcal{R}}$ as follows. Label each side $\mathcal{R}$ by the generator, say g, which pairs it by writing the label g inside $\mathcal{R}$ and $\overline{g}$ on the outside, where $\overline{g} \in \Gamma_{\mathcal{R}}$ denotes the inverse of g. Let N be the images of the boundary of $\mathcal{R}$, which we denote by $\partial\mathcal{R}$, under all elements in Γ. We can label each oriented side in N following the same rules we used to label the sides of $\mathcal{R}$. Therefore, if $g, h \in \Gamma$ and $g \cdot \mathcal{R}$ intersects $h \cdot \mathcal{R}$ along a side s, then $e = g^{-1}h$ is an element belonging to the generator set $\Gamma_{\mathcal{R}}$ and the side of s which is interior to $g \cdot \mathcal{R}$ is labelled $\overline{e}$ and the side of s interior to $h \cdot \mathcal{R}$ is labelled e.

It is not a restriction to assume that the origin 0 belongs to $\mathcal{R}$ and that no element in Γ fixes it. Therefore, we can assume that $\mathcal{R}$ and Γ satisfy these hypotheses. We construct now the Cayley graph N^* of Γ with respect to $\Gamma_{\mathcal{R}}$. Actually, the procedure we are just about to describe is an embedding of the Cayley graph in $\mathbb{D}$, but this is enough for the purposes we proposed to get in the introduction. Define

$$\Gamma \cdot 0 = \{z \in \mathbb{D} : z = g0, \text{ for some } g \in \Gamma\}.$$

Given two points $g0, h0 \in \Gamma \cdot 0$, we join them by a geodesic segment if $g^{-1}h \in \Gamma_{\mathcal{R}}$. Then label the edge from $g0$ to $h0$ by $g^{-1}h$. Carrying out this to all points in $\Gamma \cdot 0$ we get a directed graph N^* which is dual to N. Note that a path in N^* corresponds to an element in the group written in the generators in $\Gamma_{\mathcal{R}}$. If we define the length of a path as the number of edges it contains, then we can regard N^* as a path metric space where the distance between two points is the least long path that connects the two given points. We denote $|g|$ the distance between the origin 0 and the point $g0$, this is also known as the word metric of g with regard to the generating set $\Gamma_{\mathcal{R}}$

The idea of assigning a label to each side of $\mathcal{R}$ makes it possible to code oriented geodesic arcs γ in $\mathbb{D}$ which do not intersect a vertex of $\mathcal{R}$. Indeed, if γ is such an arc, then it intersects adjacent copies of $\mathcal{R}$, say $g_1\mathcal{R}, g_2\mathcal{R}, \ldots, g_n\mathcal{R}$, and then we can associate to it the element $e_1 \cdots e_{n-1} \in \Gamma$, where $e_i = g_i^{-1}g_{i+1}$. We call this element the cutting sequence of γ. If γ^* is a path in the Cayley graph N^* such that it corresponds to the cutting sequence of some geodesic arc γ, then we call γ^* a geodesic path.

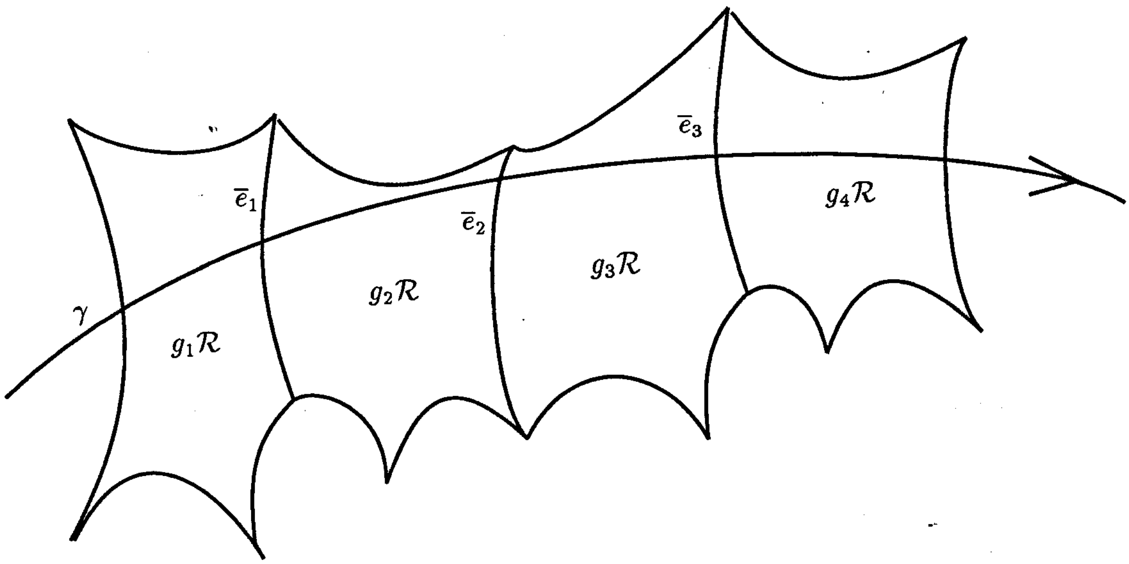

Figure 3.1. Cutting sequence. As the geodesic arc γ cuts the copies of $\mathcal{R}$, we form its cutting sequence reading off the sequence of labels corresponding to each side it intersects

The topics we have been exposing so far are certainly true for any finitely generated Fuchsian group. We now start to impose some restrictions on the groups we deal with in order to guarantee the validity of the results we will state. From now on we assume that there exists a fundamental domain for Γ with at least five sides and having the property that the net N of images of $\partial\mathcal{R}$ is a complete union of geodesics. In other words, Γ has a fundamental domain with the so called *even corners property*, see Definition 1.2.1.

Example 3.2.2 (Surface groups). If Γ is the fundamental group of a closed surface of genus $g \geq 2$, then it is possible to determine a fundamental domain $\mathcal{R}$ for Γ with the even corners property. Choose $2g$-geodesic loops as in the picture below.

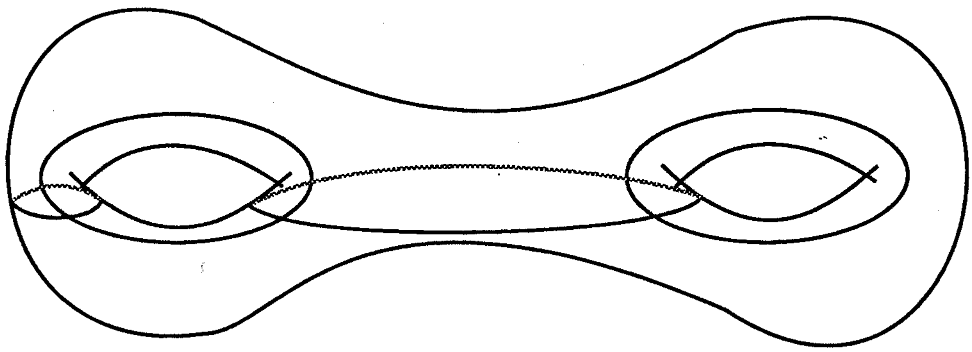

Figure 3.2. Surface of genus 2. Cutting this surface along the indicated loops renders an even cornered fundamental domain

If we cut the surface along these loops, we get a fundamental domain with $8g-4$ sides. Observe that the extension of any two geodesics having a common intersection point is still a side of $\mathcal{R}$, therefore $\mathcal{R}$ is even cornered.

The even corners property is a crucial hypotheses in order to prove the veracity of the two following lemmas.

Lemma 3.2.3. *A geodesic lying in N intersects a geodesic path in N^* at most once.*

Proof of 3.2.3.
This is Lemma 2.2, [BS87]

$\boxed{3.2.3}$

Lemma 3.2.4. *Let s and s' be two non-adjacent sides of a fundamental polygon $\mathcal{R}$ with the even corners property. Then provided that $\mathcal{R}$ is not a triangle, the geodesics which extends s and s' do not intersect.*

Proof of 3.2.4.
This is Lemma 2.2, [BS79]

$\boxed{3.2.4}$

3.3. Markov partitions and the word problem

The purpose of this section is two-fold. The first is to describe a construction which originally appeared in the work of Bowen and Series [BS79]. Their main result was to get a Markov partition for the limit set of Fuchsian groups and an expanding map associated to this partition. The even corners condition on the fundamental polygon first appeared in this work. Via their construction, it was possible to transfer important aspects of the action of the group on points in the limit set to the framework of symbolic dynamics and ergodic theory.

The second objective of this section is to further explore the connections of the the results in [BS79] and results established by C. Series in her works about the word problem for Fuchsian groups, see [Ser81], [Ser86] and [Ser91]. It is this interaction of the symbolic dynamics and the word problem that we will analyse and put it in the modern language of automatic groups.

We now develop the symbolic coding mentioned in last two paragraphs. The idea is to partition $\mathbb{D}$ into sets whose interiors are disjoint, then use the side pairing transformations to define a mapping on each element of the partition. Let $H(e)$ be the half-space in $\mathbb{D}$ not meeting $int\,\mathcal{R}$ such that its geodesic boundary contains the side of $\mathcal{R}$ whose exterior label is e. We use these half-spaces to define a transformation $f : \mathbb{D} \to \mathbb{D}$ as follows. Any point in $\mathbb{D}$ belongs to at most two of the half-planes $H(e)$ at the same time, this is a consequence of Lemma 3.2.4. Therefore, if $z \in H(e_i)$ and z also belongs to another half-plane, say $H(e_j)$, then choose $e_k \in \{e_i, e_j\}$ and define $f(z) = e_k^{-1}z$ for these points and for those points not lying in the intersection, define $f(z) = e_i^{-1}z$. Define $f(z) = z$ if $z \in \mathcal{R}$. The transformation defined in the last paragraph can certainly be extended to points on $\partial\mathbb{D} = S^1$. Indeed, prolong every geodesic in the net N which intersects $\mathcal{R}$ in either a vertex or a side until it hits the boundary at infinity, ie. S^1. Let $\mathcal{E}$ be the set of all end points of these geodesics. Given two points ξ_1 and ξ_2 in $\mathcal{E}$, we say that they are adjacent if the open interval (ξ_1, ξ_2) in S^1 does not intersect any point in the set $\mathcal{E}$. Clearly the set $\mathcal{M}$ of all closed intervals in S^1 whose end points are in $\mathcal{E}$ and are adjacent is a covering for S^1, and apart from end points we can say that $\mathcal{M}$ is a partition for S^1. Any interval in $\mathcal{M}$ belongs to the intersection of some half-plane $H(e_i)$, $e_i \in \Gamma_{\mathcal{R}}$, with S^1. We then use the same procedure we followed in the definition of f in order to get a function on S^1, which we still denote by f. One of the key properties of this function is that the image of an interval in $\mathcal{M}$ is a union of intervals which also belong to the partition $\mathcal{M}$. This is known as the Markov property with respect to the partition $\mathcal{M}$.

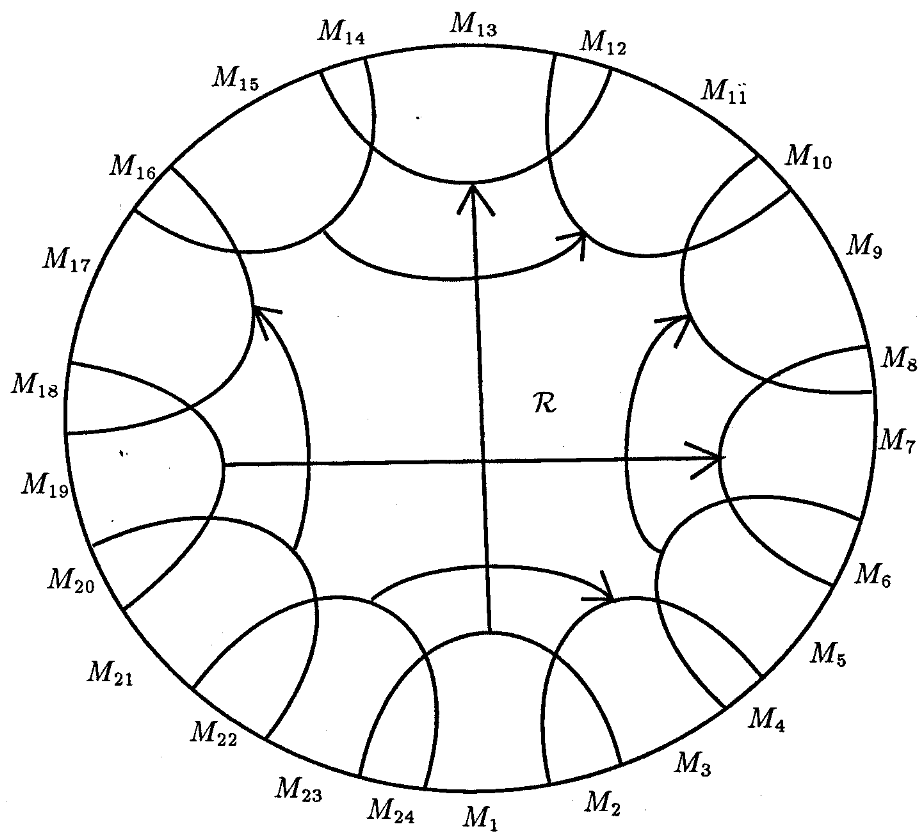

Figure 3.3. A Markov partition . In this example the Markov partition has 24 elements. The partition is obtained from the fundamental domain. Doing the indicated identifications we get a surface of genus 2. See also example 3.2.2.

We call the reader's attention to the fact that the set $\mathcal{M}$ is finite if $\mathcal{R}$ has no vertices on S^1, which is the same as the group Γ having no parabolic elements. If $\mathcal{R}$ has a vertex on S^1, then there exists a countable number of intervals in the partition $\mathcal{M}$.

The connection between the Markov transformation we have just defined and the word problem is given by a theorem due to C. Series. Before we state the theorem we need more notation.

Consider the following sets:

- $B(e) = \{z \in \mathbb{D} : f(z) = e^{-1}z\}$
- $I(e) = B(e) \cap \partial\mathbb{D} = B(e) \cap S^1$
- $B(e_0 \cdots e_n) = \cap_{r=0}^{n} f^{-r} B(e_r)$

- $I(e_0 \cdots e_n) = B(e_0 \cdots e_n) \cap S^1 = \cap_{r=0}^{n} f^{-r} I(e_r)$.

In the theorem that follows, we denote the interior of the set $B(e_0 \cdots e_n)$ by *Int* $B(e_0 \cdots e_n)$. Also, the Euclidean diameter of the interval $I(e_0 \cdots e_n)$ is written as diam $I(e_0 \cdots e_n)$.

Theorem 3.3.1.

(a) *Int* $B(e_0, \cdots, e_n) \neq \varnothing \Leftrightarrow$ *Int* $I(e_0 \cdots e_n) \neq \varnothing$.

(b) *Int* $B(e_0 \cdots e_n) \neq \varnothing \Rightarrow e_0 \cdots e_n$ *is shortest.*

(c) *If* $f(g \cdot \mathcal{R}) = h \cdot \mathcal{R}$, *then* $|h| = |g| - 1$ ($|\cdot|$ *denotes the word length norm with respect to the generating set* $\Gamma_{\mathcal{R}}$).

(d) *Each* $g \in \Gamma$ *has a unique representation of the form* $g = e_0 \cdots e_n$, *for some* $e_0 \cdots e_n$ *with Int* $B(e_0 \cdots e_n) \neq \varnothing$.

(e) *diam* $I(e_0 \cdots e_n) \to 0$ *as* $n \to \infty$.

Proof of 3.3.1.
This is Theorem 5.10 of [Ser91]

$\boxed{3.3.1}$

From now on, we shall concentrate our analysis on Fuchsian groups without cusps, that is to say, no parabolics. As we already mentioned, the Markov partition we get from the fundamental domain of these groups has a finite number of elements. Let $\mathcal{M} = \{M_1, \ldots, M_k\}$ be this partition.

Consider the subshift of finite type

$$\Sigma_f = \{x = (x_n)_{n=0}^{\infty} \in \prod_{n \geq 0} \{M_1, \ldots, M_k\} : f\, x_n \supseteq x_{n+1} \text{ for all } n \geq 0\}$$

with shift map

$$\sigma : \Sigma_f \to \Sigma_f$$
$$(x_n)_{n=0}^{\infty} \mapsto (x_{n+1})_{n=0}^{\infty}.$$

The reader should consult [PP90] for the theory of shifts of finite type.

A finite sequence written in the symbols $\{M_1, \ldots, M_k\}$ which is a block of a sequence in the shift Σ_f is called an admissible sequence. Given an admissible sequence $M_{i_0} M_{i_1} \cdots M_{i_n}$ we can associate to it a word $e_{i_0} e_{i_1} \cdots e_{i_n}$ in the group using the map $\omega : \mathcal{M} \to \Gamma_{\mathcal{R}}$ defined by $\omega(M_i) = e$ where $M_i \subseteq I(e)$. Observe that the words we get are in the form given by Theorem 3.3.1 (d).

Indeed, suppose that $M_{i_0}M_{i_1}M_{i_2}$ is an admissible sequence, such that $\omega(M_{i_k}) = e_{i_k}$, $k = 0, 1, 2,$. The condition of being admissible translates as

$$fM_{i_k} \supseteq M_{i_{k+1}}, \; k = 0, 1, 2.$$

We then have that

$$\begin{aligned} M_{i_0} &\supseteq e_{i_0}M_{i_1} \\ M_{i_1} &\supseteq e_{i_1}M_{i_2} \end{aligned}$$

and therefore

$$\begin{aligned} I(e_{i_0}) &\supseteq e_{i_0}e_{i_1}M_{i_2} \\ e_{i_0}I(e_{i_1}) &\supseteq e_{i_0}e_{i_1}M_{i_2} \\ e_{i_0}e_{i_1}I(e_{i_2}) &\supseteq e_{i_0}e_{i_1}M_{i_2} \end{aligned}$$

which implies that $Int\ I(e_{i_0}e_{i_1}e_{i_2}) \neq \varnothing$, and finally Theorem 3.3.1(a),(d) guarantees that $g = e_{i_0}e_{i_1}e_{i_2} \in \Gamma$ is such that $|g| = 3$ and it is written in the unique form provided by the Theorem.

The natural next step to take is to ask ourselves if we can determine a procedure to list all the possible admissible sequences. If so, then we will have a very efficient way of writing elements in the group. This is the main objective of the next section

3.4. Automatic structures and shifts of finite type

Our aim in this section is to interpret the results stated in the previous section within the language of automatic groups theory. More precisely, we will provide a proof that the groups we are dealing with are automatic using the shift of finite type we described. We now introduce notation and definitions concerning the idea of a finite state automaton. The reader should consult [ECH$^+$92] for the relevant aspects of automata theory related to automatic groups.

A set A with finitely many elements is called an *alphabet* and the elements in A are called *letters*. A *word* is an integer $n \geq 0$ and a map $\{1, \ldots, n\} \to A$. For the sake of simplicity, we write a word putting the images of the function one after the other starting with the image of 1 and ending with the image of n. The integer n is termed the length of the word. We denote by A^* the set of all words. The juxtaposition operation makes the set A^* a free monoid. A *language* is a subset of A^*. The languages we shall be interested in are the ones called *regular languages.* These can be defined in terms of finite state automata.

Definition 3.4.1 (Finite state automaton). A finite state automaton $\mathcal{A}$ is a quintuple (S, Y, A, μ, s_0) such that

(a) S is a finite set of *states*,

(b) Y is a subset of S, called the set of *accept states*,

(c) A is a finite set, the *alphabet*, whose elements are called *letters*,

(d) $\mu : S \times A \to S$ is a function called the *transition function*,

(e) $s_0 \in S$ is called the *initial state* or *start state.*

A finite state automaton $\mathcal{A}$ can be thought as a machine which reads a tape from left to right. The tape is divided into a finite number of squares, such that in each square there is a letter from the alphabet A. The machine is initially in the state s_0 and it reads the first letter in the tape, then it goes to another state which is determined by the transition function μ and the letter it just read. After the last letter is read the machine is in a state $s \in S$. If s belongs to the set of accept states, then we say that the word written in the tape is accepted by $\mathcal{A}$. Otherwise, the word is rejected.

The language $L(\mathcal{A})$ accepted by $\mathcal{A}$ is the set of all words accepted by $\mathcal{A}$.

Definition 3.4.2 (Regular language). A language L is called regular if it is accepted by a finite state automaton.

We now have all the necessary information to define an automatic group. We will use the definition given by Cannon in his work [Can91]. This is not the standard definition. This last one can be found in [ECH$^+$92], and the former one usually is given as a result which follows from the standard definition. More precisely, Cannon's definition appears as Lemma 2.32(lipschitz property) in [ECH$^+$92].

Definition 3.4.3 (Automatic group). Let Γ be a group and let A denote a finite set of semigroup generators for Γ. Let $\mathcal{A}$ be a finite state automaton with alphabet A and suppose that the natural map $\pi : L(\mathcal{A}) \to \Gamma$ which takes each word to the element it represents is surjective. Then we say that $\mathcal{A}$ defines an automatic structure on Γ if the following condition is satisfied. There exists a number K with the property that whenever two words w_1 and

w_2 accepted by $\mathcal{A}$ differ from one another by multiplication on the right by a single element of A, then the corresponding paths in the Cayley graph N^* are a uniform distance less than K apart.

Our objective from now on is to construct an automaton $\mathcal{A}_{\mathcal{M}}$ with the same properties as the one which is used in the first part of Definition 3.4.3. As promised in the introduction, we will base our approach on the results regarding the symbolic dynamics we discussed in the last section.

Recall that Γ is a finitely generated Fuchsian group without parabolics. We assume the existence of an even corners fundamental domain $\mathcal{R}$ for Γ, and we use the side pairing transformations to get a symmetric system of generators $\Gamma_{\mathcal{R}}$. We showed in section 3.3 how to construct a partition $\mathcal{M} = \{M_1, \ldots, M_k\}$ of S^1 and a transformation $f : S^1 \to S^1$ with the Markov property. Bearing in mind Definition 3.4.1, we define an automaton $\mathcal{A}$ as follows. The set of states S consists of all elements in the partition $\mathcal{M}$, an additional state F, which denotes failure, and one more state $*$ which denotes the start state. In short terms

$$S = \{M_1, \ldots, M_k, F, *\}.$$

The alphabet A is the generating set $\Gamma_{\mathcal{R}}$. The transition function $\mu : S \times A \to S$ is defined as

$$\mu(M_j, a^{-1}) = \begin{cases} M_i & \text{if } f|_{M_i} = a \text{ and } fM_i \supseteq M_j \\ F & \text{otherwise.} \end{cases}$$

For all $a \in A$ we set $\mu(F, a) = F$. We allow the map μ to be multivalued at the start state $*$, namely $\mu(*, a^{-1}) = M_{i_1}, \ldots, M_{i_l}$ if $\omega(M_{i_1}) = \omega(M_{i_2}) = \ldots = \omega(M_{i_l}) = a^{-1}$. We justify this assumption in the remark that follows. As the set Y of accept states we take the elements in the partition $\mathcal{M}$ plus the start state.

Remark 3.4.4 (start state). It is quite clear from the way we define the image of the transition function on the elements of the partition $\mathcal{M}$ that we want to produce admissible sequences in the shift Σ_f. If we do not allow μ to have many values at the start state $*$, then we would not allow the occurence of certain admissible sequences, and therefore we might not get a surjection from the language accepted by the automaton to group elements. The reader can see this immediately in the Markov partition given in Figure 1.3. If we set f restricted to M_1 equal to f restricted to M_{24}, then there exists an admissible sequence in two digits which starts with the symbol M_{24} but if we substitute this last one by the symbol M_1, this sequence is no longer admissible. Therefore, if we have chosen for each generator a unique element of the partition, such that the map f restricted to it is the given generator, we would miss some group elements when performing the operation which associates to an admissible sequence an element in Γ, as shown at the end of the previous section.

The language accepted by the automaton $\mathcal{A}_{\mathcal{M}}$ is the set of all words in the inverses of the generators which are shortest representatives of elements in the group, and no two words are mapped under the natural map to the same element. Indeed, suppose that $w = e_1^{-1}e_2^{-1}\cdots e_n^{-1}$ is an accepted word. Let us say that as the automaton "reads" w it goes from the start state $*$ to states $M_{i_1}, M_{i_2}, \ldots, M_{i_n}$. This means that in the shift Σ_f we have that $M_{i_n}M_{i_{n-1}}\cdots M_{i_1}$ is an admissible sequence, therefore $e_n\cdots e_1$ is a group element written in its unique form given by Theorem 3.3.1(d).

As we said before, Definition 3.4.3 has its roots in J. Cannon's ideas which are described in his paper [Can84]. He considered discrete groups G of isometries of hyperbolic n-dimensional space $\mathbb{H}^n$ acting discretely and cocompactly on $\mathbb{H}^n$. The key point in his work, which relates our discussion here, is the introduction of *cone types.* This made possible the construction of an automaton whose properties later appeared in the definition of an automatic group. Suppose that G_0 is a set of semigroup generators for G. Given a vertex x in the Cayley graph N^* of G with respect to G_0, denote by $C(x)$ the set of all vertices y in N^* which have a geodesic from the the identity vertex to y passing through x. Cannon called $C(x)$ the cone at x. Given two cones $C(x), C(y)$ one defines an equivalence relation $C(x) \sim C(y)$ if $yx^{-1}C(x) = C(y)$. An equivalence class $[C(x)]$ is called a cone type. The striking result proved by Cannon is that there are only finitely many cone types. One very important consequence of this fact is that all hyperbolic groups in the sense of Gromov are automatic, see [Gro87].

Cannon's result on the number of cone types alows us to say that there exists a finite procedure to describe the behaviour of geodesics in the Cayley graph as they go from the origin to infinity. Indeed, we can define a finite state automaton $\mathcal{C} = (S', A', \mu', Y', s_0')$ based on these observation. The set of states S' is the set of cone types of G plus an additional state F interpreted as failure, the alphabet A' is the set of generators G_0. The transition function

$\mu' : S' \times A' \to S'$ is defined as follows

$$\mu'([C(x)], a) = \begin{cases} [C(xa)] & \text{if } xa \in C(x) \\ F & \text{otherwise} \end{cases}$$

and $\mu'(F, a) = F$ for all $a \in A'$. The elements in the set of accepted states Y' are all cone types and the initial state is the cone type of the identity element in G. The language accepted by this automaton is the set of all words in G_0 which are shortest representatives of elements in the group G.

Unfortunately this automaton is very hard to construct in practice due to the dificulty in finding the set of all cone types. On the other hand, the automaton we get using the subshift of finite type Σ_f definitely has an explicit description. This is certainly the case for surface groups. We saw in Example 3.2.2 that these groups have a fundamental polygon with all internal angles equal to $\pi/2$ and with $8g - 4$ sides, where $g \geq 2$ is the genus of the surface. It is immediate to see that the Markov partition we get has $16g - 8$ intervals. If we use the regular $4g$-gon fundamental domain, we get a Markov partition with $(4g - 2)4g$ elements. So, in the case of a surface group it is possible to determine the number of states the automaton $\mathcal{A}_\mathcal{M}$ has and this is certainly a good feature of our construction. We must also say that another good characteristic of $\mathcal{A}_\mathcal{M}$ is that the natural map $\pi : \mathcal{L}(\mathcal{A}_\mathcal{M}) \to \Gamma$ is a bijection.

Observe that Definition 3.4.3 has two parts. We showed that the groups we are considering satisfy the first one, now we prove that they also satisfy the second one. Our approach will follow closely the ideas of J. Birman and C. Series in their joint work [BS87]. We now set ourselves to prove

Proposition 3.4.5. *Let w_1 and w_2 be two shortest paths in N^* with the same initial points and such that their end points are at unit distance apart. Then the distance between w_1 and w_2 is at most* 1.

This proposition finalises our proof that the group Γ is automatic.

Proof of 3.4.5.

We argue by contradiction. If the distance between the two paths is greater than one, then there must be a copy $\tilde{\mathcal{R}}$ of the fundamental domain $\mathcal{R}$ lying strictly in between w_1 and w_2.

Claim 3.4.6. *There exists a geodesic in*

$$N = \bigcup_{g \in \Gamma} g \cdot \partial \mathcal{R}$$

which intersects w_1 or w_2 twice.

Proof of 3.4.6. Suppose that the claim is not true. Define

$$\Psi = \{\psi \in N : \psi \text{ contains a side of } \tilde{\mathcal{R}}\}.$$

Let ψ_1 be the geodesic in Ψ which intersects w_1 closest to its initial point and let $\psi_2 \in \Psi$ be the geodesic which intersects w_1 closest to its end point. Denote by s_1 and s_2 the sides of $\tilde{\mathcal{R}}$ that ψ_1 and ψ_2 contains respectively. It follows that $s_1 \cap s_2$ is not empty, that is to say, they meet at a vertex of $\tilde{\mathcal{R}}$. To verify this last assertion first recall that $\tilde{\mathcal{R}}$ has at least five sides. Observe that if s_1 and s_2 do not intersect then, with only one exception, any side not adjacent to s_1 and s_2 must cut either one of these sides twice due to the choice we made for s_1 and s_2. See picture below.

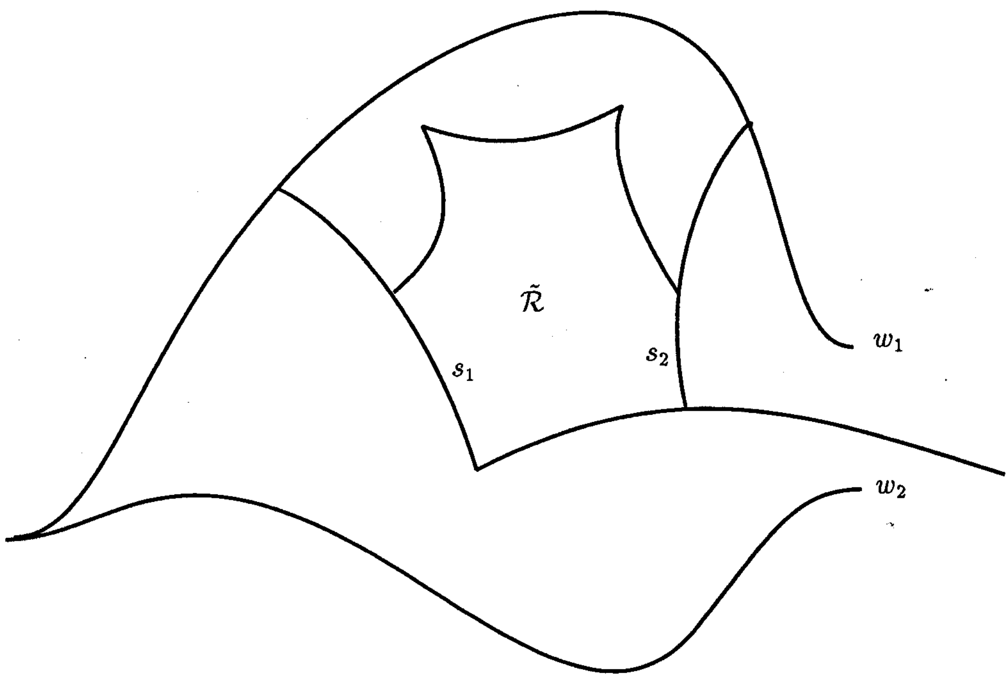

Figure 3.4. A situation which we will prove to be impossible. The sides s_1 and s_2 must inteserct

To finish the claim, use Lemma 3.2.4 to conclude that any side not adjacent to s_1 and s_2 must intersect w_1 or w_2 twice.

$\boxed{3.4.6}$

Let ψ be a geodesic in N such that it intersects w_1 twice with the property that the segment it determines in w_1 has minimal length among all the geodesics which satisfy the same condition. Denote by w the path contained in w_1 such that, apart from the end points, it has its vertices between the intersection points of ψ with w_1.

Claim 3.4.7. *Apart from its end points, the vertices of w are all the images $g0$ of 0 such that $g\mathcal{R}$ intersects ψ on the same side that w is situated.*

Proof of 3.4.7.

Any copy $g\mathcal{R}$ of the fundamental domain $\mathcal{R}$ which intersects ψ in between these two intersection points and on the same side that w lies with respect to ψ, must have $g0$ as a vertex of w otherwise any geodesic in N which contains a side of $g\mathcal{R}$ and does not intesect ψ would cut w_1 in two points which would determine a segment shorter than w and this is impossible by the choice we made for it.

Let $h0$ be a vertex of w such that $h\mathcal{R}$ does intersect ψ. Take a vertex v of $h\mathcal{R}$ which is in the region bounded by w and ψ. The extension of the two sides of $h\mathcal{R}$ whose intersection is v must cut w and ψ, this implies that we have a geodesic triangle in N which is impossible since $\mathcal{R}$ has at least five sides. $\boxed{3.4.7}$

Finally, Claim 3.4.7 implies that w_1 intersects twice a geodesic in N and then it can not possibly be a shortest path by means of Lemma 3.2.3. We therefore reach a contradiction.

$\boxed{3.4.5}$

Appendix

The purpose of the appendix is to give a proof of the following theorem.

Theorem .0.8. *Let Γ be a purely loxodromic, geometrically finite Kleinian group which is formed from Γ_1 and Γ_2 by Klein combination. Suppose that each Γ_j is quasiconformally conjugate to a Kleinian group Γ_j^0, where Γ_j^0 has the even corners property. Then, Γ is quasiconformally conjugate to a purely loxodromic, geometrically finite Kleinian group Γ^0 with the even corners property.*

We remind the reader Klein's combination theorem.

Theorem .0.9. *(see* [Mas88]*) Let Γ_1 and Γ_2 be Kleinian groups. Suppose there exist fundamental domains D_j for the action of Γ_j on $\Omega(\Gamma_j)$ so that D_1 contains the complement of D_2 in $\overline{\mathbb{C}}$ and D_2 contains the complement of D_1 in $\overline{\mathbb{C}}$. Then, the group $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is a Kleinian group isomorphic to $\Gamma_1 * \Gamma_2$, and $D = D_1 \cap D_2$ is a fundamental domain for the action of Γ on $\Omega(\Gamma)$.*

One way in which this result is used is the following argument, given in [AM77]. Let Γ be a purely loxodromic, finitely generated Kleinian group, and suppose that $\Omega(\Gamma)$ contains a component Δ which is not simply connected. Then, the planarity theorem [Mas88] implies that there exists a simple closed geodesic c in Δ which is precisely invariant under the identity in Γ. Recall that a set $X \subset \overline{\mathbb{C}}$ is *precisely invariant* under a subgroup Φ of Γ if every element of Φ keeps X invariant and if $\gamma(X)$ is disjoint from X for all $\gamma \in \Gamma - \Phi$.

Let P_1 and P_2 be the components of $\overline{\mathbb{C}} - \Gamma(c)$ which contain c in their boundaries; the P_j are well defined, as the spherical translates of any sequence of distinct translates of c must go to zero. Let $\Gamma_j = \mathrm{st}_\Gamma(P_j)$. There are two cases.

First, it may be that P_1 and P_2 are inequivalent under Γ, which we refer to as the *separating* case. Let B_j be the closed disc determined by c which is disjoint from P_j; then, B_j is precisely invariant under the identity in Γ_j. Let

D_j be a fundamental domain for Γ_j which contains B_j, and note that each D_j contains the complement of the other. The Klein combination theorem gives that $\langle \Gamma_1, \Gamma_2 \rangle$ is a Kleinian group isomorphic to $\Gamma_1 * \Gamma_2$, and that $D = D_1 \cap D_2$ is a fundamental domain for Γ. It is an easy inductive argument (see the proof of Lemma 2 of [AM77] that $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$.

In this case, we can describe somewhat explicitly the ordinary set of Γ. Let Δ_j be the component of $\Omega(\Gamma_j)$ containing B_j. Any component of $\Omega(\Gamma_j)$ which is a not translate of Δ_j will be a component of Γ. The other components of $\Omega(\Gamma)$ are obtained as follows. Let $\Phi_j = \mathrm{st}_{\Gamma_j}(\Delta_j)$, and let X_j be the complement of $\Phi_j(\mathrm{int}(B_j))$ in Δ_j. Then, the open set Δ which is the union of all translates of $X_1 \cup X_2$ by elements of $\Phi = \langle \Phi_1, \Phi_2 \rangle$ is a component of Γ, whose stabilizer in Γ in exactly Φ. Hence, every component of Γ is either a translate of an unmarked component of one of the Γ_j or is a translate of a component constructed from the two marked components.

The other case is that there exists some $\xi \in \Gamma$ so that $\xi(P_1) = P_2$, so that both c and $\xi(c)$ lie in the boundary of P_2; we refer to this case as the *nonseparating* case. Let $\Gamma_2 = \mathrm{st}_\Gamma(P_2)$ and $\Gamma_1 = \langle \xi \rangle$. Let B_1 be the closed disc determined by c which is disjoint from P_2, let B_2 be the closed disc determined by $\xi(c)$ which is disjoint from P_2, and let A be the complement of the interiors of B_1 and B_2 in $\overline{\mathbb{C}}$. Let D_2 be a fundamental domain for Γ_2 which contains both B_1 and B_2, and let D_1 be a fundamental domain for Γ_1 which contains A. Then, Klein combination implies that $\langle \Gamma_1, \Gamma_2 \rangle$ is a Kleinian group isomorphic to $\Gamma_1 * \Gamma_2$, and that $D = D_1 \cap D_2$ is a fundamental domain for Γ. It is an easy inductive argument (see the proof of Lemma 3 of [AM77]) that $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$.

Here again, we have a moderately explicit description of the components of $\Omega(\Gamma)$. Let Δ_j be the component of $\Omega(\Gamma_2)$ containing B_j, and note that Δ_1 and Δ_2 need not be distinct components of Γ_2. Any component of $\Omega(\Gamma)$ which is not a translate of either Δ_1 or Δ_2 will be a component of $\Omega(\Gamma)$. For the other components of $\Omega(\Gamma)$, we give a description for the case that $\Delta_1 = \Delta_2$; the description in the case that they are distinct is similar. Let $\Phi_2 = \mathrm{st}_{\Gamma_2}(\Delta_2)$, and let Y be the complement of $\Phi_2(\mathrm{int}(B_1) \cup \mathrm{int}(B_2))$ in Δ_2. Then, the open set Δ which is the union of the translates of Y by elements of $\Phi = \langle \Phi_2, \xi \rangle$ is a component of Γ, whose stabilizer in Γ is exactly Φ. Hence, every component of Γ is either a translate of an unmarked component of Γ_2 or is a translate of a component constructed from the marked component(s) of Γ_2.

In either case, the choice of the curve c marks components of Γ_1 and Γ_2. In the separating case, the marked component of Γ_j is the component Δ_j containing B_j. In the nonseparating case, the marked components of Γ_2 are the components Δ_1 and Δ_2 containing B_1 and B_2, respectively; note that, in this case, Δ_1 and Δ_2 need not be distinct.

We now show that it is possible to construct quasiconformal deformations of Γ by starting with quasiconformal deformations Γ_j^0 of Γ_j and a knowledge of which components of Γ_j are marked. Let $\omega_j : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the quasiconformal map inducing the deformation Γ_j^0 of Γ_j. Conjugation by ω_j induces an isomorphism $\rho_j : \Gamma_j \to \Gamma_j^0$, where $\rho_j(\gamma) = \omega_j \cdot \gamma \cdot (\omega_j)^{-1}$.

The main tools we use is the following results of Marden and of Maskit.

Theorem .0.10 (Marden [Mar74]). *Let Γ and Γ^0 be purely loxodromic, geometrically finite Kleinian groups. Suppose there exists an orientation preserving, quasiconformal homeomorphism $h : \Omega(\Gamma) \to \Omega(\Gamma^0)$ which induces an isomorphism $\rho : \Gamma \to \Gamma^0$. Then, h has a quasiconformal extension to $\overline{\mathbb{C}}$.*

Recall that a *function group* is a finitely generated Kleinian group Φ whose ordinary set contains a component Δ which is invariant under the action of Φ. If Γ is a finitely generated Kleinian group and Δ is any component of $\Omega(\Gamma)$, then $\Phi = \mathrm{st}_\Gamma(\Delta)$ is a function group with invariant component Δ; this is a good example to keep in mind.

Theorem .0.11 (Maskit [Mas77]). *Let Φ and Φ' be purely loxodromic, geometrically finite function groups with invariant components Δ and Δ', respectively, and let $\rho : \Phi \to \Phi'$ be an isomorphism. Then, there exists an orientation preserving, quasiconformal homeomorphism $f : \Delta \to \Delta'$ so that $\rho(\phi) = f \cdot \phi \cdot f^{-1}$ for all $\phi \in \Phi$.*

Using these two results, it suffices to show that there exists an isomorphism between Γ and Γ^0 which preserves component subgroups.

Consider first the separating case. Let Δ_j be the marked component of $\Omega(\Gamma_j)$ and let $\Delta_j' = \omega_j(\Delta_j)$. Suppose there exists a simple closed curve c' in $\overline{\mathbb{C}}$ so that the closed discs B_1' and B_2' determined by c' are contained in Δ_1' and Δ_2' respectively, and so that B_j' is precisely invariant under the identity in Γ_j^0. Let D_j' be a fundamental domain containing B_j'.

Klein combination yields that $\Gamma^0 = \langle \Gamma_1^0, \Gamma_2^0 \rangle$ is a Kleinian group isomorphic to $\Gamma_1^0 * \Gamma_2^0$ and $D' = D_1' \cap D_2'$ is a fundamental domain for Γ^0. Define an isomorphism $\rho : \Gamma \to \Gamma^0$ by $\rho(\gamma) = \rho_j(\gamma)$ if $\gamma \in \Gamma_j$, and then extending to all of Γ. Let Δ be a component of Γ; then, Δ is either a translate of an unmarked component of Γ_j (that is, not a translate of Δ_j) or is a translate of a component constructed from the two marked components.

Suppose Δ is a translate of an unmarked component of one of the Δ_j, and let $\Phi = \mathrm{st}_\Gamma(\Delta)$. Then, a conjugate of Φ stabilizes an unmarked component of one of the Δ_j, and so a conjugate of $\rho(\Phi)$ stabilizes an unmarked component of one of the Γ_j^0, and so $\rho(\Phi)$ stabilizes a component of Γ^0.

Suppose, on the other hand, that Δ is a translate of a component constructed from the two marked components, and let $\Phi = \mathrm{st}_\Gamma(\Delta)$. Then, Φ is conjugate to a component constructed from the two marked components of the Γ_j, and so a conjugate of $\rho(\Phi)$ is the stabilizer of a component of Γ^0 constructed from the two marked components of the Γ_j^0, and so $\rho(\Phi)$ is the stabilizer of a translate of a component of Γ^0.

So, ρ preserves component subgroups. Combining the Theorems of Marden and Maskit above, we see that ρ is induced by a quasiconformal homeomorphism of $\overline{\mathbb{C}}$, and we are done.

Similarly, in the nonseparating case we have that the isomorphism $\rho : \Gamma \to \Gamma^0$ constructed from the deformation of Γ_2 is induced by a quasiconformal homeomorphism of $\overline{\mathbb{C}}$.

We collect the above argument in the following Theorem.

Theorem .0.12. *Let Γ be a purely loxodromic, geometrically finite Kleinian group which is formed from Γ_1 and Γ_2 by Klein combination. Suppose that Γ_j^0 is a quasiconformal deformation of Γ_j, and that Γ^0 is formed from Γ_1^0 and Γ_2^0 by Klein combination in a way which preserves the marked components. Then, Γ^0 is a quasiconformal deformation of Γ.*

As an application of Theorem 2.1.2, we have a proof of Theorem .0.8.

Bibliography

[AB60] L. Ahlfors and L. Bers. Riemann's mapping theorem for variable metrics. *Ann. of Math.*, 72:385–404, 1960.

[Ahl81] L. Ahlfors. *Möbius Transformations in Several Dimensions*. Ordway Professorship Lectures in Mathematics, University of Minnesota, 1981.

[AM77] W. Abikoff and B. Maskit. Geometric decomposition of kleinian groups. *Am. J. of Math*, 99:687–697, 1977.

[AZ94] K. Astala and M. Zinsmeister. Holomorphic families of quasi-fuchsian groups. *Ergod. Th. & Dynam. Sys.*, 14:207–212, 1994.

[Ber70] L. Bers. Spaces of kleinian groups. In J. Horváth, editor, *Several Complex Variables I, Lecture Notes in Mathematics 155*, pages 9–34. Springer-Verlag, Berlin, 1970.

[Bou93] M. Bourdon. *Actions quasi-convexes d'un groupe hyperbolique, flot géodésique*. PhD thesis, Université de Paris Sud, Centre d'Orsay, Février 1993.

[Bow75] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphims*. Lecture Notes in Mathematics. Vol. 470. Springer Verlag, Berlin-Heidelberg-New York, 1975.

[Bow79] R. Bowen. Hausdorff dimension of quasi-circles. *I.H.E.S., Publ. Math.*, 50:1–25, 1979.

[BS79] R. Bowen and C. Series. Markov maps associated to Fuchsian groups. *I.H.E.S., Publ. Math.*, 50:153–170, 1979.

[BS87] J. S. Birman and C. Series. Dehn's algorithm revisited, with applications to simple curves on surfaces. In S.M. Gersten and John R. Stallings, editors, *Combinatorial Group Theory and Topology*, volume 111 of *Annals of mathematics studies*, pages 451–478. Princeton University Press, Princeton N.J., 1987.

[Can84] J. W. Cannon. The combinatorial structure of cocompact discrete hyperbolic groups. *Geometriae Dedicata*, 16:123–148, 1984.

[Can91] J. W. Cannon. The theory of negatively curved spaces and groups. In M. Keane T. Bedford and C. Series, editors, *Ergodic Theory, Symbolic Dynamics and Hyperbolic Geometry*, pages 315–369. Oxford University Press, Oxford, 1991.

[ECH+92] David Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston. *Word Processing in Groups*. Jones and Bartlett, Boston, 1992.

[Flo80] W. Floyd. Group completions and limit sets of Kleinian groups. *Invent. Math.*, 57:205–218, 1980.

[Fri86] D.. Fried. The zeta functions of Ruelle and Selberg. *Ann. Sci. E.N.S.*, 19:491–517, 1986.

[Gro56] A. Grothendieck. La théorie de Fredholm. *Bull. Soc. Math. France*, 84:319–384, 1956.

[Gro87] M. Gromov. Hyperbolic groups. In S.M. Gersten, editor, *Essays in group theory*, volume 8 of *MSRI Publ.*, pages 75–263. Springer, New York, 1987.

[KKPW89] A. Katok, G. Knieper, M. Pollicott, and H. Weiss. Differentiability and analyticity of topological entropy for Anosov and geodesic flows. *Invent. Math.*, 98:581–597, 1989.

[Lim81] E. Lima. *Curso de Análise, vol. 2.* IMPA, Rio de Janeiro, 1981.

[Mar74] A. Marden. The geometry of finitely generated kleinian groups. *Ann. Math.*, 99:383–462, 1974.

[Mas71a] B. Maskit. On poincaré's theorem for fundamental polygons. *Advances Math.*, **7**:219–230, 1971.

[Mas71b] B. Maskit. Self-maps of kleinian groups. *Am. J. of Math*, 93:840–856, 1971.

[Mas77] B. Maskit. Isomorphisms of function groups. *J. d'Analyse Math.*, **32**:63–82, 1977.

[Mas88] B. Maskit. *Kleinian groups.* Springer Verlag, New York, 1988.

[May91a] D. Mayer. Continued fractions and related transformations. In M. Keane T. Bedford and C. Series, editors, *Ergodic Theory, Sym-*

bolic Dynamics and Hyperbolic Geometry, pages 175–222. Oxford University Press, Oxford, 1991.

[May91b] D. Mayer. The thermodynamic formalism approach to Selberg's zeta function for $PSL(2,\mathbb{Z})$. *Bull. Amer. Math. Soc.*, 25:55–60, 1991.

[Pol86] M. Pollicott. Meromorphic extensions of generalized zeta functions. *Invent. Math.*, 85:147–164, 1986.

[Pol91] M. Pollicott. Some applications of the thermodynamic formalism to manifolds of constant negative curvature. *Adv. in Math.*, 85:161–192, 1991.

[Pol94] M. Pollicott. The Picard group, closed geodesics and zeta functions. *Trans. Amer. Math. Soc.*, 344:857–872, 1994.

[PP90] W. Parry and M. Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Asterisque*, 187-88:1–268, 1990.

[PS94] M. Pollicott and R. Sharp. Comparison theorems and orbit counting in hyperbolic geometry. *Preprint*, 1994.

[Roc] A. Rocha. Meromorphic extension of the Selberg zeta function via thermodynamic formalism. *Math. Proc. Cam. Phyl. Soc.*, To appear.

[Rue76] D. Ruelle. Zeta-functions for expanding maps and Anosov flows. *Invent. Math.*, 34:231–242, 1976.

[Rue82] D. Ruelle. Repellers for real analytic maps. *Ergod. Th. & Dynam. Sys.*, 2:99–107, 1982.

[Ser81] C. Series. The infinite word problem and limit sets in Fuchsian groups. *Ergod. Th. & and Dynam. Sys*, 1:337–360, 1981.

[Ser86] C. Series. Geometrical Markov coding on surfaces of constant negative curvature. *Ergod. Th. & and Dynam. Sys*, 6:601–625, 1986.

[Ser91] C. Series. Geometrical methods of symbolic dynamics. In M. Keane T. Bedford and C. Series, editors, *Ergodic Theory, Symbolic Dynamics and Hyperbolic Geometry*, pages 125–151. Oxford University Press, Oxford, 1991.

[Shi89] B. Shiffman. Separate analyticity and Hartogs theorems. *Indiana Uni. Math. Journal*, 38:943–957, 1989.

[Sul83] D. Sullivan. Conformal dynamical systems. In J. Palis Jr., editor, *Geometric Dynamics, Lecture Notes in Mathematics 1007*, pages 725–752. Springer-Verlag, Berlin, 1983.

[Sul84] D. Sullivan. Entropy, Hausdorff measures and Kleinian groups. *Acta Math.*, 153:259–277, 1984.